Low Dimensional Topology Notes July 11, 2006

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1 Gabai

[Manolescu's talk will be in this room; the other talk will be in Coalition 3.]

I'd like to give a complete proof today of the tame ends theorem, at least in the context of a hyperbolic manifold without any parabolics. This gives the heart of the story. So there are no rank one or rank two cusps.

So last time we gave a proof of Canary's theorem. Because of certain technical issues, we need to prove a certain proposition:

Proposition 1 *Let M be an irreducible homotopy handlebody and $\gamma_1, \gamma_2, \ldots$ pairwise disjoint locally finite simple closed curves not homotopic to a point map. After passing to a subsequence and allowing γ_1 to have multiple components, there exists an irreducible open W, both π_1 and H_1 injective in M and exhausted by $W_1 \subset W_2 \subset \cdots$ such that*

1. *if $\Gamma_i = \gamma_1 \cup \cdots \cup \gamma_i$ then $\Gamma_i \subset W_i$, with δW_i connected and 2-incompressible relative to Γ_i.*
2. *There exists a core D for W equal to B^3 with some one-handles (a thickened graph) so that Γ_i can be homotoped into D via a homotopy in W_i.*

Assuming M is free is not a big stretch, because we saw in general it's a finitely presented free group free product with finitely many surface groups.

A little point here is that the W has a finitely presented fundamental group so it really does have a core.

If I had fifty minutes I'd give you a pretty good exposition of this, but I don't. The proof is based on the theory of end reduction developed by Brin-Thickston, from the 1980s. As an addendum, if M is hyperbolic and γ_i are simple geodesics, then each W_i are atoroidal, meaning there are no embedded π_1-injective tori. This is a little black box. In ten minutes I could explain this.

Exercise 1 *Any torus in M is a tube (bounds a solid torus) or a convolutube (bounds a cube with knotted hole).*

If you use that fact you can prove the atoroidality.

The next black box is this:

Theorem 1 *Thurston tameness theorem*
Let W be a compact $\chi(W) < 0$ irreducible atoroidal 3-manifold, with $\delta W \neq \emptyset$. If $\hat{W} \to W$ is a cover such that $\pi_1(\hat{W})$ is finitely generated, then $int(\hat{W})$ is topologically tame.

If you just apply Thurston's hyperbolization to this, then $W = \mathbb{H}^3/\Gamma$ where Γ is geometrically finite. Subgroups of geometrically finite groups are geometrically finite. So covers corresponding to atoroidal 3-manifolds corresponding to finitely generated π_1 are topologically tame.

Corollary 1 *Let $\hat{W}_i$ denote the $\pi_1(D)$ cover of W_i then $int(\hat{W}_i)$ is topologically tame, where D is the core of W.*

There's an issue here. The cover has boundary coming from the boundary of the surface. Thurston's theorem doesn't say that the manifold compactification with boundary is tame. So one problem is to show that the two compactifications agree.

Also, we should maybe be able to find a topological proof of Thurston's tameness theorem. We need this in the context that the fundamental group is a free group.

Now the question is, we want to show this manifold is tame. To do this we show that it satisfies the taming criterion. Say it has one end. Then the end is either geometrically finite or there is a sequence of geodesics exiting the manifold. We can assume these are simple closed curves bounding tubes of uniform radius. Remember, the taming criterion says there exists a sequence of surfaces T_i mapping into N such that they exit, their genus is boundoed above, they homologically seperate, they're $Cat(-1)$. Okay, let me show you how to find these. Okay. So here's $W_{10,000}$. So what we do, given N exiting geodesics we use the topological proposition to produce the W. Then how do you find a genus three surface? Here are the words. Pass to the $\pi_1(D)$ cover of W_i. Then each δ_j has a canonical lift, fixing a basepoint. plus many other lifts.

So $\hat{\hat{S}}_i$ is the manifold boundary of $int(\hat{W}_i)$ pushed slightly into $int(\hat{W}_i)$. We need this to be 2-incompressible. So if it's not 2-incompressible, we just compress it until it is. Then $\hat{S}_i$ is this surface maximally $0,1$ compressed.

Now we do what we did in the context of Dick Canary's theorem. This might have many components. We let S_i be the component bounding Bag_i. Then S_i is chosen such that there exists p with for each $i, \ldots, \hat{\delta}_p \subset Bag_i$ and $\lim p(i) \to \infty$ where $i, \ldots, \delta_{p(i)} \in Bag_i$.

Just think of it this way. Imagine you have infinitely many marbles and then like ten buckets. You look at the first marble and put it into one bucket. If you look at the first i marbles, you just partition them into the ten buckets. Then if you look at the first j marbles, you just put them into the ten bucket. After passing to a subsequence, they'll all contain the pth marble for some p, and that bucket, the first bucket, second bucket, third bucket, will contain arbitrarily many marbles. The geodesics are like the marbles and the compressed bags are like the buckets.

So you pass to the covering space, which compactifies, you push the boundary in and do the compressions to get the surface which always contains a lifted geodesic δ_p and also geodesics of high index.

This is just purely topologically at this point. Then we take S_i and shrinkwrap it with respect to $\hat{\delta}_i \in Bag_i$. This produces a surface P_i, which gives us the T_i we want by projecting down from the cover.

There's this technical problem, that the shrinkwrapping may not want to occur in $\hat{W}_i$.

The original solution (Calegari-Gabai) was to first shrinkwrap δW_i in N. If the shrinkwrapped $\delta W_i \cap \Delta_i = \emptyset$, then $\delta \hat{W}_i$ is smooth, with mean curvature zero, so acts as a barrier for shrinkwrapping in $\hat{W}_i$. In the general case we do a limit argument. We perturb the metric near these geodesics, and take the limit as the perturbations approach the original one.

Let me show you Soma's solution, which uses a very clever covering space argument. If you want to do PL-shrinkwrapping, you need a convexity property in the covering space. But Soma noticed that this cover embeds in a branched cover of the original N. Given $p : \hat{W}_i \to W_i$, we can restrict to be away from the geodesics Δ_i. So in particular $W_i - \Delta_i$ π_1 injects into $N - \Delta_i$ since δW_i π_1 injects into $N - int(W_i)$.

Now let $X_i \to N - \Delta_i$ denote the cover corresponding to $p_* \pi_1(\hat{W}_i - p^{-1}(\Delta_i))$. Then $\hat{W}_i - p^{-1}(\Delta_i)$ embeds in this cover. Let $\bar{Y}_i$ be the metric completion. Then $\hat{Y}_i \to N$ is a branched cover over Δ_i and $\hat{W}_i$ embeds in $\bar{Y}_i$.

Lemma 1 *Bowditch*
For all j, $\hat{\delta}_j$ is the only closed curve preimage of δ_j in $\bar{Y}_i$. All the other preimages are lines. Therefore the covering space here embeds in this other larger covering space. Now we can do the PL-shrinkwrapping of S_i.

There can be infinite branching, where those parts can be, say, out here, so, and when you shrinkwrap with respect to the i geodesics on the inside and the others on the outside, it's possible that this guy will want to touch one of these guys on the outside, but that's okay. We can shrinkwrap there, as I said.

So now we need to show that these surfaces satisfy the taming criterion. The thing to do is to focus on the picture. So that surface was shrinkwrapped to P_i and then projected to T_i. We have to pass to this subsequence so now I have $T_{i_1}, T_{i_2}, \ldots$

We need to show

1. $\text{genus}(T_i) \leq \text{genus}(\delta(\text{core}))$
2. T_i is $Cat(-1)$.
3. T_i exit the manifold.
4. T_i homologically seperate.

We have the first two by consturction.

Let α_i be a locally finite collection of proper rays in N such that for all j, α_j starts at δ_j. Then if $j \leq i$ let $\hat{\alpha}_j^i$ denote the lift of α_j to $\bar{Y}_i$ starting at $\hat{\delta}_j$. Since $\hat{\delta}_p(i) \in Bag_i$, $\langle S_i, \hat{\alpha}_j^i \rangle = 1$ where $j = p(i)$. So $P_i \cap \hat{\alpha}_j^i \neq \emptyset$ because of the way the shrinkwrapping works. So when you project downstairs, $T_i \cap \alpha_j \neq \emptyset$, so that T_i hits the ray itself. That is, for all i, $T_i \cap \alpha_{p(i)} \neq \emptyset$, and $p(i) \to \infty$ as $i \to \infty$. Then by the bounded diameter lemma, these surfaces have to go off to ∞.

For i very large, $d(T_i, \delta_p)$ is large, so $[T_i] = n[\delta(\text{Core}(N))] \in H_2(N - int(\text{Core}(N)))$ where $n = \langle \alpha_p, T_i \rangle$. This is because while you generally have to go along a ray from the core to see this, for sufficiently large i, you can choose a short path from the core to δ_p. So here we only need to count intersection number with the ray α_p.

How do we make this calculation? We notice this one to one correspondence between intersection points of α_p and T_i downstairs and preimages of the ray with the surface P_i upstairs. Note that since T_i is far from δ_p downstairs, we have that $q^{-1}(\delta_p)$ and P_i are far from one another upstairs. Notice that there's one canonical lift of α_i, corresponding to starting in $\hat{\delta}_p$. I claim that $\langle \hat{\alpha}_p, P_i \rangle = 1$ if $\hat{\alpha}_p$ is this canonical lift of α_p. Since $\hat{delta}_p \subset Bag_i$, then $\langle \hat{\alpha}_p, S_i \rangle = 1$. But since $S_i \cong P_i$ by a homotopy missing $\hat{\delta}_p$, you also have $\langle \hat{\alpha}_p, P_i \rangle = 1$.

With the other lifts, the claim is that we have algebraic intersection number zero. The other preimages of δ_p are lines. So the preimages of α_p start on the lins. So then we can move these α_p along the line. Remember that S_i is homologically trivial, so that P_i is homologically trivial, meaning there's a compact chain with P_i as boundary. So we just take the endpoints and push them out to avoid the three-chain. So $\langle \beta, P_i \rangle = \langle \beta_1, P_i \rangle = \langle \beta_1, \emptyset \rangle = 0$. So when we're calculating the homology class of T_i, we get one 1 and all the others zero. So T_i represents the generator in the end. So we found the sequence of simplicial hyperbolic surfaces that seperate, exit, and have this bounded genus.

Any questions? Anyway, that's it for these lectures, thank you for your attention.

2 Fintushel-Stern

I'm again professor Fintushel-Stern. It's good pedagogy to repeat what you said before. I wanted to compare techniques useful in three dimensions in dimension four. The second thing that Ron did, he gave a user's guide to using invariants to distinguish the manifolds we're discovering. The two constructions I want to talk about today, there's one called the log transform. It's nothing more than Dehn surgery, removing and resewing $T^2 \times D^2$. So say $T \hookrightarrow X^4, X_\theta = X \backslash N(T) \cup_\theta T \times D^2$, and here θ is the gluing map. To record the information for the gluing map, you get:
$\theta : \delta(T^2 \times D^2) \to \delta(X \backslash N_T)$ So really the information that's important is homological, just like in Dehn surgery. So in T^3 you choose a basis $\alpha, \beta, [\delta D^2]$. Then we can look at three integers, as we have $\theta_*[\delta D^2] \subset H_1(\delta(X \backslash N_T), \mathbb{Z})$ and $\theta_*([\delta D^2]) = p\alpha + q\beta + r[\delta D^2]$ and we call the resulting manifold $X(p, q, r)$.

So okay, another operation, like knot surgery. So you remove $S^1 \times (S^1 \times D)$ and sew in a knot complement. I want not to disturb the homeomorphism type of the manifold. Oh, that reminds me that tomorrow at 1:00, Peter Teichner will be giving a talk about topological four-manifolds, here.

I need $S^3 \backslash K$ to have its fundamental group killed. If I can assume that the generators of the homology die in the complement, then by Van Kampen I haven't disrupted simple connectivity. So assume $\pi_1(X \backslash T) = 0$. By Alexander duality, then, this torus is homologically essential, so that $[T] \neq 0$ in $H_2(X)$. I'm not assuming the torus for the log transform is homologically essential, but I need that for the knot surgery.

Exercise 2 *Show that* $X_K \cong_{\text{homeo}} X$.

Okay, so $X_K = X \#_T S^1 \times M_K$ Now I can do zero frame surgery on the knot $K \hookrightarrow S^3$. Then I can call the manifold M_K the manifold with zero frame surgery. Now sitting inside of $S^1 \times M_K$ is S^1 cross the meridian. So this is again a fiber sum. This is just a convenient way. That's a review of the first lecture.

The second lecture was developing invariants to see if these operations were fruitful. We'll treat the Seiberg-Witten invariants very formally for the purposes of this lecture. They're defined on homology elements which are characteristic, $SW_X = \{c \in H_2(X) | cx = x^2 \mod 2 \forall x\} \to \mathbb{Z}$. Ron wrapped this up into a Laurent polynomial. Since the Seiberg-Witten invariant of a class is $\pm$ the invariant of minus the class, so we can make it symmetric.

So $SW_X \in \mathbb{Z}[H_2(X)]$ is $SW_X(0)$, which only exists if X is even, plus $\sum SW(B)(t_B + (-1)^{\chi(X)} t_B^{-1})$.

So we want to work out gluing operations, and how they affect the Seiberg Witten invariants. Ron used the theorem, suppose $X = X_1 \cup_{T^3} X_2$ and there exists a homology class $om\bar{e}ga \in H_2(X)$ that restricts nontrivially to T^3. Then $SW_X = (j_1)_*(SW_{X_1})(j_2)_*(SW_{X_2})$. Okay. So, let's suppose $0 \neq [T] \in H_2(X)$ and I do a log transform.

Exercise 3 *Check that the hypotheses of Taubes' theorem are satisfied*

So then the theorem tells us that $SW_{X_\theta} = (j_1)_* SW(X\backslash_{N(T)}) SW(T^2 \times D^2)$.

So $SW_{E(n)} = (t_F - t_F^{-1})^{n-2}$. Then $SW_{E(n)_\theta} = E(n)_\theta = E(n)_{(p,q,r)} = E(n)_{(0,1,r)}$ so p and q are sort of extraneous.

So let T_r be the "core" torus in $E(n)_\theta$. So this is a primitive homology class, but $F = rT_r$. You have to think about that. So if I let t be in the group ring of $H_2(X)$ corresponding to T_r, then I get $SW_{E(n)_\theta} = (t^r - t^{-r})^{n-1} \frac{1}{t - t^{-1}}$. If I do two of these I don't do anything bad, but if you do three log transforms you'll disrupt simple connectivity.

This is sort of a proof that the elliptic surfaces log transforms are all the same, but different for different r. Really the name of the game here is to do obvious things, and hope that someone intelligent in analysis comes up with something, and then you do your tricks.

How in the world are we going to compute the Seiberg-Witten invariant for when you sew in the complement of a knot. So $S^1 \times D^2$ is the complement of what knot? Let's see who's awake? The figure eight. No, the unknot. Okay, how do you go from this to another knot? You change crossings. You have some crossing and we want to change the crossing. What you want to do, now, is cross that with a circle. That's a nice observation, so far useless, but how do we change a crossing? The only operations I told you about are fibered sum and Dehn surgery. Make the observation, if I want to get there, that's the same thing as the other crossing, but with a Dehn surgery on a loop, this one in this picture, it's a -1 Dehn surgery. The effect of doing a -1 surgery, that wraps into a -1 right handed twist. So that changes the crossing. If this is the first time you've seen that, it's your exercise for this afternoon.

So what I'm more interested in is the complement of the knot. So I'm looking at $S^1 \times S^3\backslash$ this picture. So what other property do you have with this curve? This curve has linking number 0 so that it's nulhomologous in the complement of the knot. So I change K by Dehn surgeries, ± 1, on nulhomologous tori, to get the knot surgery. So X_K is related to X via a sequence of ± 1 log transforms on nulhomologous tori.

That's the topological input. I get a little depressed, I really do. So if r is ± 1, and I do this on a homologically essential tori, I don't change the Seiberg-Witten invariant. But on a nulhomologous torus, there's a massive change in the Seiberg-Witten invariants.

Theorem 2 *If $K \hookrightarrow S^3$, and $T \hookrightarrow S$ with $[T]^2 = 0$. I guess I mean $[T] \neq 0$. Then $SW_{X_K} = SW_X \Delta_K(t^2)$. So it's meaningful if you do it with something with nontrivial Alexander polynomial.*

So, an example corollary is that if the Alexander polynomial is not monic, then X_K cannot be symplectic, because the Seiberg-Witten invariant of the canonical class of a symplectic manifold is ± 1 so the Laurent polynomial is monic.

Certainly the way to prove this, last time Ron stated the gluing formula, in terms of the $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$ log transforms. So the name of the game is to just do the equation. Think of the Alexander polynomial. Dow do you compute it? You have portions of your knot K_+, K_-, and K_0, and $\Delta_{K_+} = \Delta_{K_-} = (t+t^{-1})\Delta K_0$. The game in this computation is to start building a tree, then at each branch in the tree the K_- becomes your new K_-. So then at the end you get some split links and unknots. Then you go back up through the tree and get the Alexander polynomial of the knot. That's why knot theory is so attractive to undergraduates.

So, let me draw the trefoil.

$$\begin{tikzcd}
 & \text{trefoil} \arrow[dl] \arrow[dr, "t+t^{-1}"'] & & \\
\text{unknot} & & \text{Hopf link} \arrow[dl] \arrow[dr, "t+t^{-1}"'] & \\
 & \text{unknot} & & 2-\text{unlink}
\end{tikzcd}$$

So $\Delta_T = 1 + (t + t^{-1})\Delta_{K_0} = 1 + (t - t^{-1})^2$.

I'm not going to prove the theorem. It'll be in the lecture notes. But the takehome message is that the important thing is the nulhomologous torus.

Exercise 4 *Let $T \hookrightarrow X^4$ with $[T] \neq 0$ and $[T]^2 = 0$. Does every X with $b^+(X) > 1$ have such a T?*

So just about every manifold you have includes a torus. But that's not true for $b_? \leq 1$.

Exercise 5 *Let $X_n = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. Show that such a T exists in X_n if and only if $n \geq 9$. So one fact is that $\mathbb{CP}^2 \#_n \overline{\mathbb{CP}}^2$, for $n \geq 5$, there are infinitely many smooth structures on these, precisely because there's a trick to find a nulhomologous torus.*

Of course it would be a nice trick to look at a manifold and find appropriate tori. Is there a sort of homology theory, a Fukaya category of nulhomologous tori? I have absolutely no idea.

Here's a wonderful thesis problem. They all have χ an integer. Construct a manifold that has χ a half-integer that's not, say, a connect sum of manifolds. Look at $E(2)$. Inside of that is a sphere S with self intersection -2. What's the boundary of its tubular neighborhood? As an exercise, the boundary is $\mathbb{RP}^3$. This admits an orientation reversing diffeomorphism. Take $E(2) \cup_{\theta:\mathbb{RP}^3 \to \mathbb{RP}^3} -E(2)$. Exercise: χ is a half-integer. But this should be a nonstandard manifold. It's been dormant. Is this a nonstandard manifold? The standard picture for this would be $E(2)\#E(2)$. It's homeomorphic to this manifold, but is it diffeomorphic to it? If it is that, it's probably doable by someone with perseverence. If it's not diffeomorphic, it's possibly doable by someone with very high IQ points.

[Why is it $n \geq 5$?]

That's the game. It could be lower, maybe it's zero? It's hard to say, maybe $\mathbb{CP}^2$ has infinitely many smooth structures.

3 Morgan

[I had a lot of trouble today, the heat is getting to me more and more during the afternoon lectures.]

Yesterday someone asked me, where's Perelman, all we heard about was Hamilton. We need a couple of consequences of the maximum principle for tensors due to Hamilton.

Theorem 3 *(Hamilton)*
As the curvature gets large, the negative eigenvalues of the curvature operator are arbitrarily small multiples of the largest positive eigenvalue. So this is for Ricci flows on compact 3-manifolds. $R(x,t) \geq 2X(x,t)(\log(X(x,t)) + \log(1+t) - 3)$. *So if you assume the curvature is at most* ± 1 *then* $R(x,0) \geq -6$, *so that* $X = \max\{0, -V(x,t)\}$.

Exercise 6 *As* $|Rm| \to \infty$, *then* $R \to \infty$. *If the smallest eigenvalue is negative, then, well,* $\lim(v/R) \geq 0$.

The second theorem is a consequence of the strong maximum principle. Let me say what that is for the heat equation $\frac{\partial h}{\partial t} = \Delta h$. Suppose the initial conditions are $h(x,0) \geq 0$, and $h(x,0) > 0$ for some x. Then $h(x,t) > 0$ for all x and all $t > 0$.

This theorem has an analogue in higher dimensions:

Theorem 4 *(Hamilton)*
Suppose $(M, g(t))$ *is Ricci flow, and that* $Rm(x,t) \geq 0$, *and at some positive time there is a zero direction for* Rm. *Then* M^3 *locally splits as* $(\Sigma^2, h) \times (\mathbb{R}, ds^2)$ *and* Σ *has positive curvature.*

Those are the two facts we'll need later on when we start chasing the analysis.

Let me talk for just a few minutes about some elementary constructions with Ricci flow, and then talk about blowup limits.

So we have a Ricci flow $(M, g(t))$ and we rescale by $Q > 0$, so we define a new family $(M, h(t))$, where $h(t) = Qg(Q^{-1}t)$. In this Ricci flow I move much more slowly. The curvature is much smaller. If $(M, g(t))$ satisfies Ricci flow then so does $(M, h(t))$. You can also time translate $(M, h(t))$ to $(M, h(t - t_0))$.

So using these two notions we can talk about blowup limits of Ricci flows. So suppose I have a Ricci flow $(M, g(t))$, and a sequence of points (x_n, t_n) in $M \times [0, T)$ where the flow blows up. So take $Q_n = R(x_n, t_n)$. Now take a sequence of based Ricci flows $\{(M, g(t), (x_n, t_n))\}_n$. So then we shift to rescale and start at $t = 0$, so that $g_n = Q_n g(Q_n^{-1}(t - t_n))$.

I write the new Ricci flows as $(M, g_n(t), (x_n, 0))$. This is the nth Ricci flow where I've scaled up and moved the time to zero. So $R(x_n, 0) = 1$. I'm interested in studying negative time. It makes sense to talk about a smooth or C^∞ limit of these. You could just ask for a limit of the time zero slice or a flow backwards for some time, or a flow back to $-\infty$.

A time 0 slice would be $(M_\infty, g_\infty(0), (x_\infty(0))$ where M_∞ is a complete manifold. Then the pullback is [[unintelligible]] uniformly to $\psi_n : K \to (M, g_n(0), (x_n, 0))$ on compact subsets. You could ask that they pull back flow, not just a point, and then maybe ask that it goes further backward in time. You might find this counterintuitive, but because you're rescaling time, there is more and more time available for the same backward flow.

[A lot of very rapid talk.]

Now, if a blowup limit exists then it has nonnegative curvature. That follows from one of the earlier theorems. So if you have a time zero slice or a backwards flow of any size, then the Ricci flow will have nonnegative curvature.

It's not flat. In fact, at the basepoint, the scalar curvature $R(x_n, 0) = 1$.

There are two issues here. The first is, when can you construct blowup limits. The second is what they look like, whether you can classify them. If the answer to the first is yes, and the answer to the second is pretty good, then you can get an idea about finite time singularities.

Okay, so let's talk first about, what do you need to be able to construct a blowup limit?

In general, that's a hard problem. I'm asking for C^∞ convergence, so I have to control higher derivatives of the curvature. Fortunately, there's a bootstrap procedure to ratchet up bounds on the curvature to bounds on its higher derivatives.

Lemma 2 *Shee's lemma.*
If $|Rm| \le C$ then there exist constants $c_1, c_2, \ldots,$ so that $|nabla^i Rm| \le c_i / r^i$ on $B(x, t, r/2)$.

That takes care of one of the three parts of geometric limits. You need to control the curvature, its higher derivatives, and the injectivity radius.

But if we have a sequence of Ricci flows $(M_n, g_n(t), (x_n, 0))$ and if for all R then $|Rm| \le C(R)$ on $B(x_n, 0, r) \times [-\epsilon(R), 0]$, and if $inj_{(x_n,0)}(M_n, g_n(0)) \ge \delta > 0$, then there is a smooth geometric limit at time 0.

With those two conditions you get control of the derivatives with Shee's theorem nad then you can pass to a subsequence. If you had better control in the time direction, you could pull your flow back to $-a$, and if you had a sequence of these, you might be able to go back to $-\infty$.

So you need to control the curvature and injectivity radius bounded away from zero.

Let me give you an example. You find in the manifold pieces that look like $S^2 \times (-\epsilon, \epsilon)$. The metric is not the product, but it's close to the product. Varying a little bit the metric on the cylinder, it could be negative.

[What if you started with a hyperbolic manifold?]

Those will inflate. You know there won't be finite time singularities. I don't know a statement for negative curvature, even bounded away from zero. Maybe it could be so bouncy that things could bubble off.

So you must control curvature at a bounded distance and injectivity radius at the basepoint.

Theorem 5 *(Perelman)*
Given $(M^3, g(t))$ a Ricci flow, which is developing a singularity at finite time, then for any blowup sequence (x_n, t_n) along which the scalar curvature goes to ∞, there exists a subsequence with a geometric limit.
This limit is an ancient solution, meaning it exists from time $-\infty$ to 0, it is nonnegatively curved, not flat, has bounded curvature, and is what I will call κ-non-collapsed. Here κ is a fixed positive number depending on [the bound on the injectivity radius?] and upper bound of the norm of the Riemannian curvature.

Let me say what κ-non-collapsed means. We have a ball $B(x, t, r)$ and the backward parabolic neighborhood in time $-r^2$. That's $P = (x, t, r, -r^2)$. Suppose $|Rm| \leq r^-2$ on P. Then $vol(B(x, t, r)) \geq K \cdot r^n$. This is a statement in all dimensions.

This is intimately related to the question of the injectivity radius.

Suppose I have a complete manifold of bounded Riemannian curvature that is κ-collapsed. Then it depends only on the curvature bound and the lower injectivity radius bound. Bounding the volume below is like bounding the injectivity radius below.

We're going to have a control of the injectivity radius so strong, that it will pass to the limit. Then the injectivity radii are bounded away from zero. I'll also get that in the limit it will be κ-non-collapsed.

If I had to rate the originality of Perelman's contributions, this is it.

[Did he draw any inspiration from gravity?]

I don't know.

How do you get control here? You introduce the $\mathscr{L}$-function. So you have, say, (x, T). I want to look at $\gamma(\tau)$ where τ takes values as $T - t$. I have $\gamma(\tau)$ which lies in $M \times T - \tau$. So $\gamma(0) = x$. We define the $\mathscr{L}$-length of a path as $\int_0^{\bar{\tau}} (\sqrt{\tau}) R(\gamma|_T) + |X(\tau)|^2 dz$. Here $\gamma'(\tau) = (x(\tau), -1)$.

We can try to minimize the length of the paths back from (x, T). So I get $\nabla_X X = \frac{1}{2} \nabla R -$

$\frac{1}{2\tau}X - 2Ric(X, \cdot)$.

If you reparameterize with $s = \sqrt{\tau}$ this is a ODE which is nonsingular at 0.

Up in the tangent space we'll have an open set $\tilde{U}(\tau)$. At time $T - \tau$ we'll have the image $U(\tau)$. Once you know the tangent vector you get the whole differential equation. Then you get an exponential to come down. You have the open dense cut locus. Then the map on $\tilde{U}$ is a diffeomorphism. These sets vary with τ. The closure af $\tilde{U}_\tau$ is contained in $\tilde{U}(\tau')$ if $\tau' < tau$. So the τ are minimal geodesics but they may not be unique, but if you back off you get something complete and unique and the exponential map is an isomorphism.

On the cut locus, things are more complicated, but this is analogous to how you deal with this via Riemannian geometry.

All right. So how are we getting toward the injectivity radius? It turns out to be useful to study $\ell(\gamma) = \frac{1}{2\sqrt{\tau}} \int_0^T \sqrt{\tau}(R + |X|^2)d\tau$. This is scale invariant.

Now we look at the reduced volume $\tilde{V}(W \times \{T - \tau\})$ is $\int_W \tau^{-n/2} e^{-\ell(w)} dw$. I wish I could tell you what the integral means but I can't. When is it going to be small? It will be small when ℓ is large. The τ is just to make it behave well under scaling.

Theorem 6 *$\tilde{V}$ is monotone nonincreasing as a function of τ when we flow along minimal $\mathscr{L}$-geodesics.*

Here's the point x at time T, here's the space W i'm studying. I'm going to pretend that I'm in the good set, every point has a unique family of $L - geodesics$ going te it. I comupute the comparison of the two reduced volumes. I get that $\tilde{V}(W(\tau')) \geq \tilde{V}(W(\bar{\tau})$. Let me use this to produce κ-non-collapse.

Start now with $(M, g(0))$. I want to prove that this is κ-non-collapse. Come back to, say, the 1/2 level. If my original metric has curvature bounded by one, I'll have control, nothing will have gone to a singularity. Now there is a γ with $\ell(\gamma) \leq n/2$. Now take a ball centered at time 0, and draw curves from the point x_t down to the ball.

[unintelligible]

The $\ell_{B^4 \times \{0\} \times \cdots} \leq C$ and then $\tilde{V} \geq \int \tau^{-n/2} e^{-C} vol(B)$. You're only supposed to check things on the ball

[unintelligible]

The volume factor is going to zero, so you won't get much reduced volume from these geodesics, you have a fixed amount of reduced volume down here and get a fixed amount of ordinary volume here. That's the basic argument for the κ-non-collapse.

At each point in the manifold you have the curvature scale, to give a ball of radius one on which the curvature is bounded by one. You really only check this at the curvature scale.

Hamilton would always take a sequence along which the curvature was maximal. The curvature everywhere would be bounded by one. Here we could have two singularities, so you have to prove that the distance rescaled by the rescaling for one singularity, goes to infinity. I've postponed that from today until tomorrow.

I think I'd better quit, I see Kirby back there giving me the time to quit sign.

4 Kirby

[Rob is a topologist famous for triangulating manifolds, or almost triangulating manifolds, and the Kirby calculus in four dimensions.]

My title used to be "How mathematicians think in dimension four" and I was going to chase away the topologists. I was told not to do that, so then Bob Edwards suggested this topic.

To talk about Boys' surface I have to talk about the projective plane P^2. Some people talk about the complex projective plane, but this will always be the real projective plane.

Recall R^n is just real space with n coordinates $\{(x_1, \ldots, x_n)\}$. inside that is the unit ball $B^n = \{x_1^2 + \ldots + x_n^2 \leq 1\}$. The three ball is the ball you bounce, the two-ball is the disk in the plane. The one-ball is the unit interval, and the higher ones are what they are.

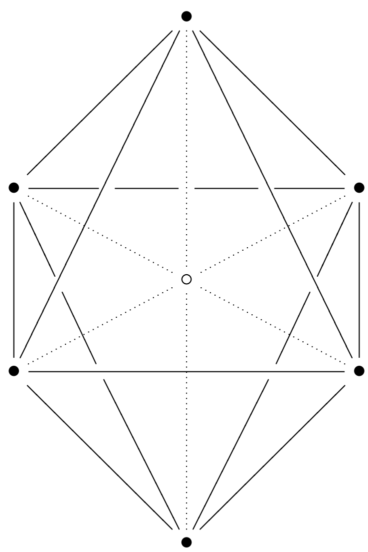
Then the sphere is the boundary of the ball $S^{n-1} = \{x_1^2 + \ldots + x_n^2 = 1\}$. So S^2 is the sphere you think of, S^1 is a circle, and S^0 is a pair of points. Now P^n is the n-sphere, and then you glue together points at opposite ends of a diameter. So it's $P^n/x \sim -x$. So P^0 will glue together the two points of S^0 into one point. For P^1, you want to glue together opposite points on the circle. Anyway, you go like that. That means, I just take away this arc and glue it on the other side. I can take this arc, which goes up there, I can cut this arc and glue it to there. Then finally there are these two points which need to be glued together, they're opposite ends of a diameter, and you can only do that by pulling them together, and so you get the circle.

So how is it that we do this with the two-sphere? You can start off with the two-sphere. One way is to take off the upper cap and the bottom cap. We'll put that cap aside. Now they're gone, we set them aside. The next thing you can do is think of the front half being glued to the back, so I can throw away the front, and am left with the back. Now one side has to be glued to the other side. But remember, it's across diameter so in particular, you flip all the points around. When you finish doing the identification after throwing away the arctic and antarctic caps, you get a Mobius strip. So now that circle has to be glued to the edge of this, which is this circle right here. You get out your needle and thread and glue the circle to the circle. There's no way to do this in three dimensions, it can't exist. That's because the projective plane does not embed, does not sit inside three-space. You've probably seen another example of this because you've probably seen pictures of the Klein bottle. This is a cylinder and the cylinder is attoched back to itself, not like this, to give an inner tube, but like this, so you pass through the side and glue to the bottom and get a Klein bottle. The point is that this does not embed. It intersects itself and it's not supposed to. In four space you could step out into fourspace and cross over and then glue it in.

It's the same for the projective plane. You could take the circle boundary in the mobius strip and unwind it in a movie, in time, and then glue in the disk. But it's hard to imagine this in three-space.

Let me say one more word about immersions. An immersion must have a continuously turning tangent. That's the property an immersion must have. This cusped picture does not have a continuously turning tangent. This is also an immersion because it has a continuously turning tangent.

In 1901 a German mathematician named Werner Boy came up with an immersion of P^2, which is hard to do. If you google this you can find lots of pictures, but they're not fully satisfactory, because they don't show what's happening on the inside. Now you might have two parts of the surface intersecting in a line, but then you might imagine that if you had a third part of the surface intersect that, it would end up with a triple point. You can prove that any immersion of P^2 must have a triple point. So we could start with that.



So when they picked a logo for PCMI they thought, "Let's see, what will Rob Kirby need for his talk in 2006?" So look at this picture of an octahedron, or like K_6 with a triple-intersection point. Here is the triple point, and you can see the three planes here which lead to them. Now you can also see three squares, the horizontal square and the three vertical squares. You add the two triangles on the sides on the top and the two triangles in the front and back on the bottom. Now I want to argue that this is the projective plane.

So first I want to argue that this is a surface. When you put two faces together, you can round that off. If you put a third face in on an edge, you wouldn't have a surface. This red edge is the boundary of this triangle and this square. The same argument is true at every edge. So that's how you check that it works at the edges. Now let's check the vertices. Up at this vertex, you have this line from this triangle, then this other triangle that we added in. Then there are the two squares that we added in. What I've drawn there looks kind of like two triangles that have met at a corner.

The two squares give the two lines that cut across one another, and the two triangles give the other two lines.

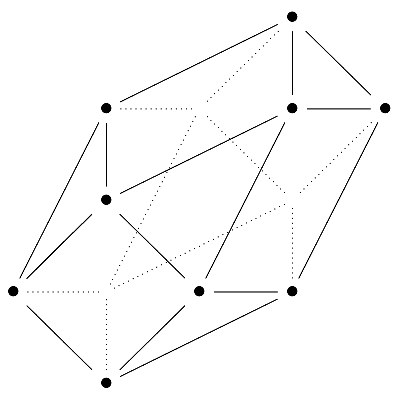
So this figure eight is an immersion of the circle. The cone of a regular circle is just a disk. The cone of the immersed circle is a disk which is not actually immersed, but it's a disk.

Now the question is whether we have an immersion, and whether it's P^2. So it's not immersed at the six vertices. It doesn't have a continuously turning tangent at the vertices. But let me convince you that this is the projective plane.

To do that let me remind you of the Euler characteristic, which is the number of vertices plus the number of faces minus the number of edges $V - E + F$. For P^2 it's 1 Why is that? Let's break up the sphere. If we break up the equator with two vertices and two edges and two faces, we get $V - E + F = 2 - 2 + 2 = 2$. So to get the projective plane we identify the two vertices, the edges, and the faces, and get $1 - 1 + 1$ and so that's an Euler characteristic of one. So we have 6 vertices, 12 edges, and 7 faces. What, you didn't expect it to come out wrong, did you?

[Is this a correct count? Maybe in the immersion, two vertices intersect so you should count them twice instead of once.]

You have to look at this to see that it's not the case. You can take, if you take the cube, there's a cube, and now chop off every corner. Cut across diagonally and remove the corners. Then you have an object that's still a sphere, the boundary is, and you can still use the antipodal map to identify opposite things. When you do the cutting, I want to take a big corner off.



Okay, so there's this problem with the six vertices where it's not immersed. So we have to do another trick.

So this is a line of intersection. Is purple visible? You have this line of intersection. You have this circle that runs from here, around, and has a double point in it. Now if you were to put your fingers on this part of the object, pick it up, and push this part down flat, you would have the cone on the figure eight. If you just lift this up, that would look like there are rays going out from the point to all of the points on the figure eight. So I could find a bunch of these in the picture I've drawn. I could come up with a line connecting these two double points in this picture, corresponding to this red line here.

So then you can draw these two together, and sort of pull them together. We want to replace these two cusps, along the red line, so that instead of having the two bad points, we connect them, push them toward each other and cancel them.

We colud do this one to get rid of those two and this one to get those two, and then get rid of these two with this edge.

The virtue is that you don't just see a pretty picture, you see how it's built. Then bits of the projective plane will intersect in curves. So you see this thing. Now you've takn this curve, and extended it to a curve that runs along there. So then you extend them and make them meet along this curve. The double curve dous this and then comes back, and comes along this side, and comes back to where it was on this side. So you have one circle of double points, with one triple point in the middle of it.

As I say, we were discussing it last Thursday or Friday, and Bob Edwards basically told me this description, Bob gavo me the mathematics, and [unintelligible]drew the picture, it's like the evening news. I should have gotten a beautiful woman out in the audience to read the news to you. I didn't do anything. To me, this is enlightening, and the other description is less so.

In the last few minutes I want to say how it's useful in turning the sphere inside out. I could never follow it and tell what's going on. I'd never seen the two-sphere turn inside out. If you draw the circle, here's the inside. Suppose I wante to move this through immersions, now the red side is poking out, but if you keep on doing this, most of the red side is out, you've almost turned it inside out, but there's a nagging problem there. You get a cusp when you pull it tight, you don't get an immersion. You can't turn the circle inside out that way, and it's a theorem of Whitney's that you can't do it at all.

Then there's this theorem of Smale in 1956 that you could take the 2-sphere through immersions inside out. The first time this was actually visualized, it was by a blind mathematician named [unintelligible].

The description that made the most sense conceptually to me uses Boy's surface. There's a perpendicular line to any point in a surface in three-space. If we take the unit interval, we cut them off, we just take the line interval everywhere. This is called the line interval bundle. Everywhere you have the interval $[-1, 1]$ on Boy's surface. So now on the outside, the -1 and 1 points, those define the sphere again. Up here there'll be a piece, don't here there'll be a piece. So everywhere along Boy's surface are these two points. That's the sphere. It's immersed. Let's say the red is inside again, notice what we can do. We can take this point and this point, taking this point north and this point south, take them through each other, and we've turned the colors inside out. When the two endpoints meet, we're at the projective plane. That's Boy's surface, an immersion of the sphere, going around twice.

There's a difficult problem I don't have anything to say about. Now you have to move that sphere through immersions to the round sphere. You might think of putting a straw in there and blowing it up but you don't know that that doesn't create cusps. In that I don't really have any insight, you might think, well, how do you unwrap that and make it round? That's your homework problem. Thank you.

[Questions?]

[Can you see a Mobius band in Boy's surface?]

Yes, but that's another homework problem.

Come on up, Bob.

[You drew a picture that reminded me of something I forgot to tell you, Rob. There's another picture that is easier to see but hides some of the symmetry.]

Here I gave a 50 minute lecture instead of a 75 minute one, but I failed, because now he'll fill up the rest of that time.

The Klein bottle can be cut in a way that will give you two mobius bands. I should be doing this in color. Over here, this is leading to another picture of Boy's surface. Here the horizontal cut circle goes like this. There's an immersed circle, and hanging down from it is the Mobius band, just the bottom half of that. This circle in the plane is regularly homotopic, that is, homotopic through immersions, to the standard circle. There drawn in the blackboard plane is the same circle. In general principles Whitney's theorem says since the turning number is one you can do it. But you can see it. Push it out over this curve. So there's a regular homotopy. Put the regular homotopy, and then cap off with a disk. The single triple point occus when th first push goes across the point there. So the homework exercise is to compare the two immersions and see that they're the same.

[How many immersions are there up to regular homotopy?]

Another homowork problem. Do you know the answer?

[I think it's one.]

[I have a question and a comment. The question is on the regular homoteopy of P^2 into $\mathbb{R}^3$, I guess that was asked.]

The smart guys in the back said two.

[The comment was, I think there was a video before.]

I think the first visualization was by [unintelligible]in the sixties which is likely to be before any video.

[Construct a map from that immersion to an embedded sphere, can you always do that?]

Yes, by Smale's theorem, but I don't know how to visualize it.

[Any other questions? Let's thank Rob again.]