Low Dimensional Topology Notes
July 10, 2006

Gabriel C. Drummond-Cole

July 12, 2006

1 Gabai

[I'd like to announce that the morning session will be for Dave and in the afternoon for Fintushel-Stern.]

Welcome back, I suppose I should turn this on. So today I want to do mainly two things. One is to show you a proof of the PL-shrinkwrapping theorem. The second is to give a detailed understanding of the proof of Canary's theorem. Everything's clear there, but the general case is very much modelled on the proof of that theorem. Modulo a couple of black boxes I think that would give a good understanding of the general situation.

Let me remind you of the

Theorem 1 *PL-Shrinkwrapping theorem*
If S is an embedded surface in a hyperbolic 3-manifold N with Δ a locally finite collection of geodesics such that S seperates Δ, $S \cap \Delta = \emptyset$, and S is 2-incompressible relative to Δ, then there exists o homotopy $S \times [0,1] \to N$ from S to a simplicial hyperbolic surface, which only touches Δ at time 1.

Recall

Lemma 1 *If α_0 is a path away from Δ from x to y then there exists a unique length minimizing piecewise geodesic α_1 path homotopic to α_0 such that if $f : I \times I \to \hat{N}$ is the homotopy, we have that the homotopy touches Δ only at time* 1.

Here $\hat{N}$ is a branched cover of N where the branching may be infinite, over geodesics through balls.

The second lemma we need is a local view of a Δ-geodesic.

Lemma 2 *If α is a Δ-geodesic, with $x \in \alpha \cap \Delta$ then either $\alpha \in \Delta$ or α near x lies in the union U of two half disks glued along Δ. Also, $\alpha|_U$ is geodesic.*

So the union of the two things is isometric to a piece of hyperbolic 2-space, wherein the curve α looks geodesic.

Exercise 1 *Prove this.*

Here's the next lemma, the cone angle lemma.

Lemma 3 *Suppose $S_0 \subset \hat{N}$ is a closed simplicial hyperbolic surface without the cone angle condition such that through each vertex passes a Δ-geodesic (lying in S) then S_0 is a genuine simplicial hyperbolic surface, that is, the cone angles are at least 2π. We assume that the implicit pushoff of the Δ-geodesics is compatible with S_0.*

We know that in the second picture of the preceding lemma, where one crosses a geodesic, the curve could not have all been on one side of the Δ-geodesic, or else it would have stayed on that time.

The hypothesis is that you can verify the cone angle condition as long as through every surface lies a Δ-geodesic.

This is similar to the interpolation theorem proof of the other day. The proof is very much similar. That is, what is the union of all the angles, and how do you make that calculation? If you see the triangle here, then from the point of view of the unit tangent bundle, the vectors pointing to the opposite edge of such a triangle, then the length of the geodesic path in the unit tangent bundle is the same as the angle measure.

So let $\delta subset S$. There's a Δ-geodesic passing through the vertex v and D in the surface a small disk around v with E a component of $D - \delta$. The cone angle contribution of E is the length of a piecewise geodesic path β in the unit S^2.

So β is a path of unit vectors which point from v to δE.

Exercise 2 *Prove that the length of β is at least π. (Hint: apply the preceding lemma)*

Two applications of the above calculation complete the proof.

Here's the last lemma we need.

Lemma 4 *Local shrinkwrapping*
Let σ be a 2-simplex with sides α, β, γ, vertex ω opposte to γ, and $f : \sigma \to \hat{N}$ with $f(\sigma) \subset B^3$ and $B^3 \cap \Delta$ is one geodesic passing through the origin. Then $f|_\alpha, f_\beta$ are Δ-geodesics, with γ a standard piece of geodesic and $f|_{\mathring{\sigma}} \cap \Delta = \emptyset$. Then the conclusion is that there's a homotopy (a Δ-homotopy on the interior, meaning it only hits Δ at time 1) relative to the boundary with the ending state of the homotopy a simplicial hyperbolic surface.

The proof has three cases.

If the triangle is disjoint from Δ you can just fill it in. If both edges hit the geodesic, you can do a local subdivision like in this picture, and if one edge hits o geodesic, you do this other local subdivision. You just have to check the cases.

. *Proof of the theorem* So first you let γ be a simple closed curve on S and then take a bunch of other curves based on γ to cut S into a disk. So first homotope the surface so that γ is a Δ-geodesic. Call the resulting surface S_1. Then homotope S_1 relative to γ so that the other arcs are Δ-geodesics.

So this process leaves you with some $4n$-gon, you can then foliate by radial segments from a vertex. Then one can homotope D relative to the boundary so that these arcs become Δ-geodesic. So we can shrinkwrap S so that the leaves of the foliation are Δ-geodesics.

Okay, so now Δ homotope S so that the 1-skeleton of a triangulation, whose edges lie in the leaves on the previous singular foliation, is Δ-geodesic. The boundary of each 2-simplex σ consists of one very short geodesics and two nearly parallel Δ-geodesics. Also the interior of σ doesn't intersect Δ. So we want these to be geodesic.

So to make this occur, we subdivide this long, skinny, triangle, subdividing it so that we can isolate the confusing parts. Each of these will satisfy the local shrinkwrapping lemma.

The only issue now is the cone angle condition. The vertices of this triangulation lie along Δ-geodesics, those from homotoping their respective curves. The new vertices created by subdivision lie along geodesic portions of the edges.

I used 2-incompressibility because, if you just have a loop linking around a geodesic only once, it will contract to a point. So the point of two-incompressibility, for applications it doesn't matter, but for a nice surface you need this condition.

[Why do you want the surface to seperate the geodesics?]

We don't always need that, it's just that, actually, for the purposes of PL-shrinkwrapping you don't need that situation. If you want to just homotope a surface to a minimal surface in the smooth case it might fly off the manifold. So, but actually, on the other hand, psychologically it's good to think there are geodesics on both sides.

Here's a central idea to tell you why shrinkwrapping is so good.

[The amount of subdivision will depend on how complicated Δ is near S?]

How much subdivision do you need? That's a good question, I don't know how to control the complexity.

You have this simple principle, that geodesics trap surfaces. If you have a path from one of the geodesics to the other, then it hits the surface with algebraic intersection number 1. If it's only allowed to touch the two geodesics at the last instant, you know that you will still have algebraic intersection number one with the same path. We have the geometric thing as well as the topological thing, the bounded diameter lemma, saying that a simplicial hyperbolic surface has bounded diameter away from the Margulis tubes. So the distance of the shrinkwrapped surface, well, modulo Margulis tubes, it lives a bounded distance from this path.

Similarly, you can imagine a ray going from a geodesic to infinity, then if the ray hits the surface algebraically once. Similarly in the result, the eventual homotoped surface will be a bounded distance from the ray.

Now let's prove Dick Canary's theorem.

Theorem 2 *If $\mathscr{E}$, an end of S is topologically tame, then it's geometrically finite or satisfies the taming criterion*

If $\mathscr{E}$ is geometrically infinite, then let δ_i be a sequence of simple geodesics exiting the end. In the special case $N = S \times \mathbb{R}$, you can shrinkwrap the cross-sections between successive geodesics. So the taming criterion is to find a sequence of $Cat(-1)$ surfaces which homologically seperate and exit of sufficiently high genus. So why do the resulting things exit and why do they homologically seperate? They exit because they trap surfaces. So $S \times 3$ seperates the third and fourth geodesic. So then the corresponding surface lies a bounded distance from a path connecting these.

Why does it homologically seperate? Homotopy preserves that property.

You ask yourself how far you have to go to contain the first, say, thousand Margulis tubes. If you've gone out distance 10000, which contained the first hundred Margulis tubes. Then the shrinkwrapped portion pops out in, it has to exit one, and follow distance one, and lie in another, and so on.

Let's just consider a handlebody, say of genus 16. We can arrange it so that the i geodesic lies between $S \times 6$ and $S \times 7$. From the point of view of the ambient manifold the surface just shrinks down. So compress it once to do compressions, and then do the 1-compressions. There may be no hope that any of the components, after this, will be far out enough or homologically seperating. So skip out to 10,000. You may more or less get the same picture, so you say, oh no, what do I do now? What you do is, what I'm about to say is a simple application of the pigeonhole principle. What you should think of, is $S_{10,000}$ is the boundary of a big bag of geodesics. When we do a compression, it's like twisting off some spots. We replace the bag by some other bags. We want to pick out the right component afterward to do shrinkwrapping. After passing to a subsequence and picking out the right component, well, first you will have a given geodesic p and second, there is a function $p(i) \to \infty$ which will lie in the bag in a fixed manner, and as the bags go to infinity, the $p(i)$ will also go to infinity.

Please accept for now that the bags will always contain the p geodesic and also some very large geodesic, and so we shrinkwrap with respect to the geodesics involved with the one-compressions te get what we would call T_i.

So how do we see that these seperate and exit? If you look at a ray $\delta_{p(i)} \to \infty$, this ray hits T_i once, so that T_i is a bounded distance from this ray. If $p(i)$ is extremely large, then the ray is far out in the manifold, and then the surface is very far out in the manifold. Therefore the surface really does exit the manifold.

The question now is why does it homologically seperate? So how do we calculate the homology class? Topologically this thing is a surface cross $\mathbb{R}$, so $H^2(\mathscr{E})$ is $\mathbb{Z}$. So to calculate the homology class, calculate the intersection number with a ray going off to ∞. So if T_i is far off in the manifold, then it's disjoint from a path from the core to δ_p. A ray from δ_p to ∞ intersected R_i once, and now it has to intersect T_i once. Since we're so far out, we intersect this ray from the core to ∞.

So we weren't really in this situation where we did any of this bad compressibility.

I explained the taming criterion, and the shrinkwrapping theorem, and now I showed you that a geometrically tame manifold satisfies the taming criterion or is geometrically finite.

So I want to give a hint for why general hyperbolic manifolds satisfy the taming criterion.

Exercise 3 *Can the manifold described by this picture be hyperbolic?*

You can't find embedded surfaces but you can find interesting surfaces that are immersed. If you stare at this picture for a long enough time you see an immersed surface of genus two which self-intersects. Here's another way of seeing this. If you look at the $\pi_1(D)$ cover, if you have a handlebody and pass to this cover, you get a manifold whose interior is topologically a handlebody of genus two, and then your ends look like a Cantor set. Then the immersed surface lifts to an embedded surface, which captures the original geodesic.

The point is that even in the topological world you can find immersed surfaces that lift to embedded surfaces in the right covering space. So inside the handlebody are these geodesics that lift to be closed geodesics. This picture looks very like the surfaces used in Canary's theorem. So we get basically a simplicial hyperbolic surface in the covering space and project it down. The claim is if we do this right we get these surfaces exiting the manifold.

Sorry to have gone ten minutes over. The goal of tomorrow is to give a proof of this modulo some black boxes. Thanks for coming.

There's a technical issue that the surface might want to exit the boundary. In the branched cover the manifold is complete. You can do the shrinkwrapping there.

2 Futer, Open books and hyperbolic Dehn surgeries

This is joint work with Gabriel Indurskis.

Disclaimer: In this talk, all 3-manifolds are compact, oriented, closed or with disjoint tori as boundary.

It's a pleasure to be here, I guess I should start with a definition, for those who aren't already familiar.

Definition 1 *An open book decomposition of M is a link $K \subset M$ called the binding and an identification where a manifold with boundary (M, K) will be $(S \times [0,1]/\sim, \delta S \times [0,1]/\sim)$ where $S = \Sigma_{g,n}$ is oriented and $\varphi : S \to S$ is a homomorphism with $\varphi|_{\delta S} = id$. Here the equivalence is $(x, 0) \sim (\varphi(x), 1)$ for all $x \in S$. and $(x, t) \sim (x, t')$ in δS.*

The question is, suppose $M - K$ is hyperbolic. Is M hyperbolic?

Why do we care? All closed M^3 have open book decompositions. So this gives all manifolds. Second, we know how to tell if $M - K$ is hyperbolic so it's a surgery problem like those Cameron was talking about. So let $M - N(K) - V_\varphi = S \times [0,1]/(x,0) \sim (\varphi(x), 1)$, which we call a punctured surface bundle over S.

Definition 2 *$\varphi : S \to S$ is called*

- *periodic if $\varphi^n = id$ for some n.*
- *reducible if φ fixes a multicurve (set of essential curves) on S.*
- *pseudo-Anosov if it is neither of the other two.*

Theorem 3 *(Thurston)*
V_φ is hyperbolic if and only if φ is pseudo-Anosov.

This is great, we know exactly how to tell when these things are hyperbolic.

I'm going to be focussing on the following examples: $\Sigma_{1,1}$ and $\Sigma_{0,4}$, the punctured torus and four punctured sphere. Then φ can be identified with an element of $SL(2, \mathbb{Z})$ uniquely up to conjugation and sign. Then φ is pseudo-Anosov if and only if φ has two distinct real eigenvalues, meaning $\varphi = \pm R^{a_1} L^{a_2} R^{a_3} \cdots L^{a_n}$ where $R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Say n is the "syllable length $\ell(\varphi)$ of φ".

So here's a question to ask. Given a hyperbolic bundle V_φ, is a particular Dehn filling hyperbolic? So if S is $\Sigma_{1,1}$ or $\Sigma_{0,4}$ and we're filling the punctures, the answer is no. You'll get respectively Solv geometry and $S^2 \times S^1$ geometry. This I'll call trivial filling and leave aside.

Another example is, if $S = \Sigma_{1,1}$, and $\varphi = RL$, then V_φ is S^3 minus the figure eight. Then you have lots of exceptional surgeries. Slightly more general than this, if S is the punctured torus, and $\varphi = \pm RL^n$ for some n, then V_φ has three Seifert fibered fillings of the form S^2 with three singular fibers. The filling of the puncture is not one of these.

A final example is that if $S = \Sigma_{1,1}$, and if $\varphi = (RL)^5 L$, then V_φ has at least two exceptional fillings.

I just want a sufficient condition for having no exceptional surgeries, rather than a classification of all exceptional surgeries.

Theorem 4 *Bleiler-Hodgson*
If $S = \Sigma_{1,1}$ then there exists an N such that if $\ell(\varphi) \geq N$, V_φ has no (nontrivial) exceptional surgery.

So the conjecture they made was that this number was twelve, since it has to be even and we have just shown an example of a length five word with exceptional surgeries.

Theorem 5 *Futer-Indurskis*
If $S = \Sigma_{1,1}$ and $\ell(\varphi) \geq 12$, then all nontrivial fillings of V_φ are hyperbolic

Theorem 6 *If $S = \Sigma_{0,4}$ and V_φ has c boundary tori (c is $1,\ 2,$ or 4). If $\ell(\varphi) \geq 3c$ then all nontrivial fillings are hyperbolic.*

[What about three boundary tori?]

This is the case I can say something about.

The method of proof involves angle triangulations. I'm going to spend the next twenty minutes or so talking about those.

An angled ideal tetrahedron T is modeled on an ideal tetrahedron in hyperbolic space $\mathbb{H}^3$. The properties of this are that the vertices are "out at infinity" and the dihedral angles, around each vertex $\alpha + \beta + \gamma = \pi$, so that opposite edges have the same dihedral angles. If you truncate an ideal vertex, you get a Euclidean triangle, and when you glue together, these will tile the boundary of your manifold.

When we glue tetrahedra together, we require that around every edge of M the dihedral angles around that edge sum to 2π. Here's the picture. Here's an edge, and here are the Euclidean triangles. These sum to 2π, and the triangles don't have to line up completely, so you don't actually have to get a hyperbolic structure. There are shearing singularities.

Now, I'm interested in hyperbolic manifolds, I want to figure out if their fillings are hyperbolic manifolds. Why would I study this gizmo which doesn't give you a hyperbolic structure?

There are several good answers; I'll only give you two.

1. They are easy to find and deform. All you have to do to make a triangulation into an angled triangulation is solve a set of linear equations and inequalities. The solution set is an open convex polytope in $\mathbb{R}^M$. To find an actual hyperbolic metric is hard. You

will get at most one solution. In practice you can only do it when you already know what you're going to get.

2. You have a lot of control over the surfaces in your manifold. If there's one takehome message from Cameron Gordon's course, it is that if you want to understand the Dehn fillings, you had better understand the surfaces in your manifold.

I guess I'd better tie this shoe before I trip over it.

Now normal surface theory says that any essential surface $(F, \delta F) \subset (M, \delta M)$ can be placed in normal form where F intersects tho tetrahedra in standard disks.

I won't list all of the standard disks, but let me give some examples of a few. One standard disk is a triangle cross section that is not incident on any edge on one face, but on all three other edges (a "triangle") or a quadrilateral which is incident on all but two opposite idges.

But if I truncate by tetrahedron there are several other types, which hit the trancated parts.

I won't write the rest of them down. They get more complicated, which I hope to convince you is a good thing.

The measure of complexity is what is called the combinatorial area,

$$A(D) = \sum_{\delta D} \underbrace{\mathscr{E}_i}_{\text{external}} + \pi|\delta D \cap \delta M| - 2\pi.$$

So $A(D)$ for a triangle or a boundary bigon is 0, while for the quadrilateral or this other picture it's α, the length of this edge.

Let me explain why this measures complexity.

Lemma 5 $A(D) \geq 0$ *with equality only for triangles and bigons.*

You can prove this by counting or by a slicker argument.

Lemma 6 *Peter Casson*
$A(F) = -2\chi(F)$ *which is like Gauss-Bonnet.*

I should really credit him for developing all of this.

It seems like I haven't done very much, but it has very powerful consequences.

Theorem 7 *(Casson, Lackenby)*
Let M be a manifold with an angled triangulation. Then M admits a hyperbolic metric.

All this basic messing around, solving linear equations and inequalities tells you that the manifold has to be hyperbolic. You don't know, say, that you can get a positively oriented triangulations or any shapes, but you do know that you can get such a metric.

By Thurston, to prove this is hyperbolic, all we have to do is rule out surfaces of negative Euler characteristic, in other words, prove M is "simple."

For a sphere and a disk, these surfaces have positive Euler characteristic, so they'd have negative area, but you don't have anything with negative area. You can have T^2, which has $A(T^2)$, but only built from triangles and bigons. But the torus is closed while bigons go out to the boundary, so it's all made of triangles. Each one of these is boundary parallel. You'll glue these together, meaning your entiere torus will be boundary parallel. Those are not essential.

Finally, you have to worry about annuli. An annulus has area zero so is built by triangles and bigons. So you can't really glue triangles to bigons because in triangles edges go from edges to edges while in bigons they go to the boundary. You have to use only bigons. So in general all the bigons do is go around an edge, and so your annulus is just a tube around an edge and we're done.

If you have geometrization, this is then pretty easy.

To get surgery information, it's a similar idea. For surgery, specifically you have the following theorem

Theorem 8 *Lackenby*
Let M be a manifold with with an angled triangulation. Let $s_1, \ldots, s_n$ be slopes on the boundary tori of M. Suppose that for any essential surface $(F, \delta F) \subset (M, \delta M)$ with these boundary slopes, $A(F) > 2\pi|\delta F|$. Then $M(s_1, \ldots, s_n)$ is irreducible, atoroidal, and not Seifert fibered.

The irreducible and atoroidal are almost immediate. If you had a torus you'd have to have a punctured torus upstairs, where you get some other bound on the area and a contradiction. For a Seifert fibered surface you have to use the ubiquity theorem of Dave Gabai.

Corollary 1 *Either by Perelman or the orbifold theorem, $M(s_1, \ldots, s_n)$ is hyperbolic.*

This is the general setup. Let me go back to our original setting and I'm going to assume that our manifold is actually a punctured torus bundle $V_\varphi = \Sigma_{1,1} \times [0,1]/\sim$. I claim that $\Sigma_{0,4}$ is very similar.

I need to tell you what the triangulation is. In this case, it was constructed by Floyd-Hatcher. It involves studying the Farey complex, also known as the curve complex of $\Sigma_{1,1}$. The vertices will correspond to essential arcs from the boundary to the boundary modulo isotopy. The edges will correspond to pairs of disjoint arcs, and the triangles to triples of disjoint arccs or ideal triangulations of S.

Let me draw the picture that makes this clear. Here's the punctured torus. Here a line from the puncture to the puncture has a well-defined slope. If you chose three different disjoint slopes, they would cut it up into a triangulation. You can draw this picture, as Saul once showed me, without lifting your chalk from the blackboard.

This is our Farey complex, it's a lovely object. Now what you have to do is look at the action of the monodromy map φ on $\mathscr{F}$. The dual to this complex is an infinite tree. There is an invariant line through the dual tree. Then just look at how this line crosses the triangles. It takes lefts and rights across Farey triangles. Here L and R are the same as the matrices I showed you earlier. So each time you cross an edge, you're changing the slope of the surface. You trade one slope for another. This is a diagonal exchange, also known as a tetrahedron. If you have a parallelogram with slopes 0 and 1 then the diagonals have slopes ∞ and $1/2$. F. Gueritaud found a really nice parameterization of the space of angles for this triangulation. There will be one parameter for every tetrahedron, and the parameters will be 0 and π. All you have to do to get an angle triangulation is that it has to be concove. Then the parameter on a hinge has to be bigger than [unintelligible].

I've seen his source code, he calls this part Chaboing because it goes along saying Chaboing, Chaboing,

We use $\pi/2$ and $\pi/2+\epsilon$. Almost all the tetrahedra are very very flat. Every time you pass a hinge tetrahedron you pick up some combinatorial area, and it turns out to be enough to get the surgery estimates. These angles are quite far from the complete structure.

I'll stop there.

[Do you think it would be possible to specify the data of an angle triangulation directly on the cusps?]

Yes, you get a picture of layered triangles on the cusps. Each tetrahedron is completely determined by the triangle it cuts off.

[How sharp are these bounds on surgery slopes?]

They are completely sharp for $\Sigma_{1,1}$, as far as the formulation goes.

[What about higher genus?]

We have some results. That's definitely where we want to go with this.

3 Fintushel-Stern

I'm the other Ron. There's been a lot of talk about how to visualize in four dimensions. I learned this technique from Dennis Johnson. He pointed out that $\mathfrak{so}(4)$ is really $\mathfrak{so}(3)\times\mathfrak{so}(3)$. So if you could move your two eyeballs independently then you could see in four dimensions. This discussion was lubricated. Since we'd already decided at a very young age how to perceive images, you couldn't change that. But the discussion went on and we thought, if you could find a baby...

Today will be a user's guide to Seiberg-Witten invariants. Without invariants your theory is worthless. Think about the bushel basket of homotopy spheres. What is it worth now? Knowing what your invariants are and how to calculate them is step one.

Let me remind you of an important cohomology class for complex surfaces. Suppose X is a complex surface with T^*X its cotangent bundle. Then an important associated line bundle is the determinant or canonical bundle. K_X. Its first Chern class $c_1(K_X) \in H^2(X, \mathbb{Z}$, which, by Poincaré duality I'll associate with a homology class which I'll also call K_X.

Here's an example. So the elliptic surfaces $E(n)$. So $E(1) = \mathbb{P}^2 \# 9\bar{\mathbb{P}}^2$. This admits a fibration structure over $\mathbb{P}^1$ whose generic fiber is a torus.

Exercise 4 *See this*

Start with a pencil of cubics and blow up the nine basepoints to see that you get a fibration.

So $E(n)$ is defined as the n-fold fiber sum of $E(1)$. Because it's a fiber sum, it will still fiber over S^2 with fiber the torus. Since it's the fiber of a fibration it will have neighborhood $F \times D^2$. So remove such a neighborhood and glue together along $F \times D^2$. So $K_{E(n)} = (n-2)F$. So as a special case, $E(2)$ is the $K3$ surface. This is one of our four manifolds. This has $K_{E(2)} = 0$.

Exercise 5 *Remember the renormalizations we did of the signature and the Euler characteristic. So* $c(E(n)) = 0$*, and* $\chi(E(n)) = n$*.*

If C is an embedded complex curve in X of genus g, then $2g - 2 = C^2 + K_X C$, which is called the adjunction formula.

Exercise 6 *Prove it.*

From one point of view, the input of gauge theory to topology gives us a way to mimic K_X in the setting of smooth manifolds.

At this point let me say a few words about Seiberg-Witten theory. Suppose we start with X a smooth closed oriented 4-manifold with a Riemannian metric.

Then the input for the Seiberg-Witten equations comprises

1. The characteristic line bundle L over X. This means the first Chern class, reduced modulo two, is $w_2(X)$ (mod 2), the second Stiefel-Whitney class of X. Another way to see this is, if $PD(c_1(L)) = k$. then $kx = x^2$ mod two for all $x \in H_2(X, \mathbb{Z})$.
2. A pair of complex two-plane bundles $W^{\pm}$ whose determinant bundles are L.

Then the Seiberg-Witten equations are PDEs whose variables (A, ψ) are, well, A is a connection on L and ψ is a section of W^+. The automorphism group of the bundle L acts on the solutions, and the quotient is called the moduli space of solutions M_L.

It turns out that as long as there is no 2-torsion in $H_1(X, \mathbb{Z})$, these equations depend only on L. We're basically interested in only simply connected manifolds, we're going to assume this. These equations really depend on a $Spin^c$ structure and a little more information. It only depends on $c_1(L)$ if there's no two-torsion in H_1.

This moduli space M_L has a formal dimension $d(L) = \frac{1}{4}(c_1(L)^2 = 3\sigma(X) + 2e(X))$.

I'll write down the equations in a special case. If you have nontrivial isotropy of the action of $Aut\ L$, the solution is called reducible. So then the variables are $(A, 0)$. In the reducible case, the equations reduce to the anti-self-duality equation $F_A^* = 0$. Without writing equations it's hard to see hawe the metric is used, but in this case we can see how the metric is used.

If there are no reducible solutions, and the moduli space is nonempty, it's a compact manifold whose dimension is the formal dimension $d(L)$. Furthermore, it can be oriented.

[Why is $d(L)$ an integer?]

It's the index of an elliptic operator.

You can orient M_L by choosing an orientation for $H^1(X, \mathbb{R}) \oplus H^2_+(X, \mathbb{R})$. When the dimension $d(L) = 0$, the moduli space gives a finite set of points with plusses and minuses associated to each. Then we can count with sign, and this gives us a number, which is our Seiberg-Witten invariant. So $SW_X(k)$ is the count of M_L with signs, where $k = PD(c_1(L))$. If the manifold is empty, then $SW_X(k) = 0$.

So SW_X gives a map from characteristic elements of $H_2(X, \mathbb{Z})$ to $\mathbb{Z}$. This depends on the choice of metric. But for generic metrics, if $b^+X \geq 1$ then there are no reducible solutions. For a generic path of metrics for $b_+X > 1$ there are no reducible solutions.

[Doesn't k depend on a choice of line bundle L?]

No. Let's take $\mathbb{CP}^2$. Then H^2 is generated by 1. Then aH is characteristic for a odd.

Okay, so SW_X is an oriented diffeomorphism invariant if $b_X^+ > 1$.

Classes k such that $SW_X(k) \neq 0$ are called basic classes.

The orientation choice will only affect the sign.

Here are some properties for $(b_X^+ > 1)$.

1. there are only finitely many basic classes.
2. $SW_X(-k) = (-1)^{\chi(X)} SW_X(k)$
3. Suppose k is a basic class and Σ an embedded closed surface of positive genus, and

$[\Sigma]^2 \geq 0$. Then $2g-2 \geq [\Sigma]^2 + |k \cdot [\Sigma]|$. This is called the adjunction inequality.

Corollary 2 *If you have an embedded $\Sigma \subset X$ with $g \geq 1$ and $[\Sigma]^2 > 2g-2$, then $SW_X \equiv 0$.*

4. if X is a symplectic 4-manifold, then $SW_X(K_X) = 1$ (Taubes)
5. if X is a minimal Kähler surface with $c_1^2(X) = c(X) > 0$, (Kähler means there exists a metric g with $(Jx, y) = \omega(x, y)$ is a symplectic form. Minimal means that there are no spheres of square -1) then the only basic classes are $\pm K_X$.
6. If X admits a metric of positive scalar curvature then $SW_X \equiv 0$.

It's useful to view $SW_X \in \mathbb{Z}H_2(X, \mathbb{Z})$. So associate to α in $H_2(X, \mathbb{Z})$ be t_α. So $t_0 = 1$.

Then $SW_X = \sum SW_X(\alpha) t_\alpha$.

If X is minimal Kähler with $c_1^2 > 0$ then $SW_X = t \pm t^{-1}$.
For $E(2)$, $SW_{E(2)}(0) = 1$.

Exercise 7 *Use adjunction inequality arguments to see that* 0 *is the only basic class of* $E(2)$*, so that* $SW_{E(2)} = 1$.

Okay, blowing up. One standard operation on four-manifolds, is to take $\overline{\mathbb{CP}}^2$. Is a negative Hopf disk bundle over S^2, union with B^4. This is a tubular neighborhood of S^2 with self intersection, of square, -1. Blowing up is replacing a ball by this, so it's just $X \mapsto X \# \overline{\mathbb{CP}}^2$. The formula for $SW_{X \# \overline{\mathbb{CP}}^2} = SW_X(e + e^{-1})$, where $e = t_E$ and E is the exceptional curve S^2 in $\overline{\mathbb{CP}}^2$. So you can say $SW_{X \# \overline{\mathbb{CP}}^2}(k \pm E) = SW_X(k)$.

So we need gluing theorems like this. So suppose $X = X_1 \cup X_2$ with $\delta X_1 = \delta X_2 = X_1 \cap X_2 \cong T^3$.

Under the extra condition that there exists $\omega \in H^2(X, \mathbb{R})$ which restricts nontrivially to δX_i, and $b^+ = 1$, closed, then X_i have Seiberg-Witten invariants, but they can be half-infinite series.

For example, $SW_{D^2 \times T^2} = \frac{1}{t^{-1} - t} = t + t^3 + t^5 + \ldots$

Taubes' gluing formula, under this hypothesis, looking at the inclusions, $SW_X = j_1 * (SW_{X_1}) j_{2*}(SW_{X_2})$. The theorem is that the result is a Laurent polynomial.

Let's do an example. If $T \subset X$ of square zero and $N_T = T \times D^2$, then $X = X \backslash N_T \cup N_T$. So $SW_X = SW_{X \backslash N_T} \cdot \frac{1}{t^{-1} - t}$. So $SW_{X \backslash N_T} = SW_X(t^{-1} - t)$.

[What did you do?]

I multiplied both sides by $t^{-1} - t$. That was the first question that reminded me of teaching at home.

Okay, so

$$1 = SW_{E(2)} = (\underbrace{SW_{E(1)\backslash N_F}}_{\pm 1})^2$$

And $E(2) = (E(1)\backslash N_F) \cup (E(1)\backslash N_F)$. So $SW_{E(1)\backslash N_F} = -1$.

So then $E(3) = E(2)\#_F E(1)$ and $SW_{E(3)} = 1 \cdot (t^{-1} - t)(-1) = t - t^{-1}$.

Exercise 8 $SW_{E(n)} = (t - t^{-1})^{n-2}$.

If the torus is not [unintelligible]then you won't get anything good from the Taubes formula.

So here's the Morgan-Mrowka-Szabo formula. It might take me the rest of the hour to write it down. It's not my fault. Oh, I guess Tom isn't here. Okay, first I have to tell you about surgery. Suppose $T^2 \times D^2$ is embedded in X. Then suppose I have a diffeomorphism $\varphi : \delta(T^2 \times D^2) \to \delta X\backslash N_T$. Then $X_\varphi = (X\backslash N_T) \cup_\varphi (T^2 \times D^2)$.

Then X_φ depends only on $\varphi_*[\delta D^2] \in H_1(\delta X\backslash N_T)$.

Okay. So choose a basis $\{\alpha, \beta, [\delta D^2]\}$ for $H^1(\delta(X\backslash N_T))$ and then $\varphi_*([\delta D^2]) = p\alpha + q\beta + r[\delta D^2]$. Then define $X_T(p,q,r) = X_\varphi$.

Then the formula is

$$\sum SW_{X_T(p,q,r)}(k_{(p,q,r)}+iT) = p\sum SW_{X_T(1,0,0)}(k_{(1,0,0)}+iT)+q\sum SW_{X_T(0,1,0)}(k_{(0,1,0)}+iT)+r\sum SW_{X_T(0,0,1)}(k$$

Here $k_{(a,b,c)}$ are those classes which restrict appropriately on X with N_T excised to the same thing to which k restricts:

$$\begin{tikzcd}
H_2(X_T(a,b,c)) \arrow[r] \arrow[dr, dotted] & H_2(X_T(a,b,c), N_T) \arrow[d, "\cong"'] \\
 & H_2(X\backslash N_T, \delta) \\
H_2(X) \arrow[r] \arrow[ur, dotted] & H_2(X, N_T) \arrow[u, "\cong"]
\end{tikzcd}$$

So $k(a,b,c)$ is anything whose image under the upper composition is the same as the image of k under the lower composition.

4 Morgan

[I've been asked to remind you that there's a switcheroo for the research talks. Lipschitz will be in here and Taylor in Silver King 1.]

I'm going to give a series of talks this week about the work of Richard Hamilton on Ricci flow and the breakthrough due to Perelman that proved geometrization and the Poincaré conjecture.

Conjecture 1 *Poincaré Conjecture*
If M^3 is closed, simply connected 3*-manifold then $M^3 \cong S^3$.*

Conjecture 2 3*-D spherical spectrum conjecture*
If M^3 is closed and $\pi_1(M)$ is finite then $M \cong S^3/\Gamma$ where $\Gamma \subset O(4)$ acting freely.

The most general of the conjectures is

Conjecture 3 *Assum M^3 is prime and orientable (the statement is more complicated for nonorientable manifolds). Then M^3 has a decomposition along incompressible tori and Klein bottles into pieces which are geometric, i.e., admit locally homogeneous Riemannian metrics of finite volume.*

The idea is to take a space, put any old metric on it, and use a parabolic flow to make the metric better. If you chose a nice metric, it will flow to one where there are these geometric pieces.

There are two things that happen here. You could assume the manifold was prime to begin with, but that's not a good differential geometric condition. The flow equations will develop singularities in finite time. So you also have to deal with the singularities.

According to Hamilton, it was Yau who suggested that these two things would counterbalance one another, and the singularities would cut the manifold apart via the prime decomposition.

Once you do this process, resolve the singularities, you get what is called Ricci flow with surgery, defined for all time.

Then in the limit at time infinity you will get the Thurston metric.

So first I'll discuss Ricci flow, and then Ricci flow with surgery. There you'll have analytic and topological control, analytic to continue this inductive process past each successive surgery, and topological to make conclusions about what happens after the singularities are resolved.

So you have a one-parameter family of manifolds M_t for $0 \leq t < \infty$. So if M_T satisfies the geometrization conjecture, then M_t also does, for $t < T$.

If you want the full geometrization conjecture, you get a bifurcation, where one piece becomes hyperbolic. In the rest of the manifold the metric collapses into very short loops. Then you use some topology, to see the collapsing conditions give graph manifolds.

I said the argument bifurcates, because you can avoid the geometrization conjecture, just proving the other two, with easier arguments. That gives

Theorem 9 *If M^3 is closed and $\pi_1(M)$ is a free product of finite groups and $\mathbb{Z}$, then M is the connected sum of space forms and $S^2 \times S^1$.*

In this case we don't have to care about the limit of the flow because:

Theorem 10 *For M as above, $M_T = \emptyset$ for some $T > 1$.*

All right, so we want to use a differential equation to deform the metric. We won't be talking about surgery for quite a while, we're going to talk, for now, about Ricci flow. We're looking for a one-parameter family of metrics on a 3-manifold $(M, g(t)), 0 \leq t < T$. So $\frac{partialg}{\partial t} = \Psi(g, \text{spatial derivatives of } g)$. If this is going to be elliptic so we can solve it, we want it to be invariant under the natural transformations of the gauge group. We know what those are, those are the curvatures.

I assume we've all had some Riemmanian geometry. If not you probably won't be able to follow this. To (M, g) you can associate a canonical connection ∇, called the Levi-Civita connection. So $X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ and the torsion free condition $\nabla_X Y = \nabla_Y X = [X, Y]$. Along with being a derivation over functions, this is well-defined; it's an easy condition to check.

So now

$$\tilde{R}(X, Y)(Z) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_[X, Y])(Z),$$

and then

$$Rm(X, Y, Z, W) = \langle \tilde{R}(X, Y)(W), Z\rangle = \langle \tilde{R}(X, Y)(Z), W\rangle.$$

I call this the Riemannian curvature tensor. The $\tilde{R}$ tensor is more natural to look at, it has less sign ambiguity, but we want the sphere to have positive curvature

So the sectional curvature of a plane is $Rm(X, Y, X, Y)$.

Okay, so what do we want to put into Ψ? We could try Riemannian curvatures. We need a tangent vector in the space of symmetric contravariant 2-tensors. The right-hand side has to be a symmetric 2-tensor. So the Riemannian curvature is not of the right structure.

So we can move to a derived tensor,

$$Ric(X, Y) = g^{ij}(Rm(X, \partial_i, F, \partial_j)) = \sum Rm(X, Z, Y, Z).$$

There's one other curvature that comes into this story, which is the constant curvature, the trace,

$$\sum_{Z \in \text{ an orthonormal basis}} Ric(Z, Z),$$

So this Ψ will have to be $aRic + bg + cR \cdot g$. It turns out that what we want is

$$\frac{\partial g}{\partial t} = -2Ric(g) = \Delta g + \text{ quadratic algebraic terms in } g$$

This is like a smoothing or correction of the heat equation $\frac{\partial h}{\partial t} = \Delta h$. Because you have the quadratic term, these will blow up in finite time. So that's what causes the problems, not the tensorality.

Okay. So Rm is a symmetric bilinear form on the wedge of the tangent bundle $\wedge^2 TM \otimes \wedge^2 TM \to \mathbb{R}$.

A basis for $\wedge^2 TM$ arises by taking an orthonormal basis $\{e_1, e_2, e_3\}$ of the tangent space and then taking the wedge of that basis $\{e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2\}$. With respect to this basis, this Rm looks like $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$. I always assume $\lambda \geq \mu \geq \nu$. For Ricci this looks like $\begin{bmatrix} \mu+\nu & 0 & 0 \\ 0 & \lambda+\nu & 0 \\ 0 & 0 & \lambda+\mu \end{bmatrix}$.

So now we have the Ricci flow equations. This isn't a strictly parabolic equation because of the action of the gauge group, but nevertheless we have

Theorem 11 *(Hamilton)*
For M^n_g compact there exists a solution to the Ricci flow equation for $(M, g(t)), 0 \leq t < T_{\max} \leq \infty$ defined for some $T_{\max}$. It is unique.

This $T_{\max}$ depends on the initial conditions.

We don't know exactly how this flow might fail, there's a lot of ways things might fail, it might stop being positive definite or the entries could blow up.

Theorem 12 *(Hamilton)*
If $T_{\max} < \infty$, then

$$\lim_{t \to T_{\max}} \max_{x \in M} |Rm(x,t)|) = \infty$$

It's not the metric but the curvature that's going bad.

Let me give you some examples. The nicest ones to understand are the Einstein manifolds, where the Ricci curvature is a multiple of the metric, $Ric(g) = \lambda g(t)$. If $\lambda = 0$ then we can all evolve that, we can just leave them constant.

In dimension four the Einstein manifolds are not classified, but

Theorem 13 *(Levi Civita)*
In 3 dimensions every Einstein manifold has constant curvature.

Let's take our favorite Einstein manifold in dimension two, S^2, with the standard metric g_{sph}

but with radius $(\sqrt{2})^{-1}$. Then $Ric = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2}g$. Then if we want

$$\frac{\partial g}{\partial t} = \lambda'(t)g = -2Ric(c) = -2Ric(g_0) = -g_0.$$

So $\lambda' = 1$ and then $\lambda(t) = (1-t)$. So as you approach $t = 1$ the metric shrinks down to a point and the curvature goes to ∞. Along the way we simply have rescaled versions of the sphere.

We see the finite time singularity, and we see that the amount of time for which the solution exists depends on the initial metric. If we start with a smaller radius, we would shrink to a point in less time.

Similar arguments apply for hyperbolic manifolds of constant negative curvature. In this case the flow exists for all time and curvature tends toward 0 as t goes to ∞.

I want to show you one other example because it's crucial for later analysis. It's not an Einstein manifold. I want to look at $S^2 \times \mathbb{R}$ with the product metric $g_0 \times ds^2$. It won't surprise you to know that the Ricci flow is just the product of the two Ricci flows.

So the flow is $(1-t)g \times ds^2$. The metric is shrinking in two of the three directions and is constant in the third. So this is not a rescaling of the original metric. So here's $S^2 \times \mathbb{R}$, and here's a smaller $S^2 \times \mathbb{R}$, and the limit as you approach 1 (in a Gromovian sense) is a line. This is an example of what is called the gradient-shrinking soliton.

The gradient shrinking soliton equation is $Ric = \lambda g$ for positive λ. So there exists an $f : M \to \mathbb{R}$ and $Hess(f) + Ric = \lambda g$. This produces metrics so that at every time there are isometries (which are not the identity) so that, well, you con get a family of diffeomorphisms $g_t : M \to M$ with $g_t^* g(t) = \lambda(t)g(0)$. This is gradient shrinking because $\lambda'(t) < 0$. Here g_t is the family of diffeomorphisms obtained by integrating $\nabla_{g(t)} f$.

The first real theorem in the subject with topological conclusions, conclusions leading to a nice topological result, is

Theorem 14 *(Hamilton)*
If (M^3, g_0) has positive Ricci curvature, then the Ricci flow $(M^3, g(t))$ develops a finite time singularity. As it does the diameter approaches 0. and $(M, diam(M, g(t))^{-2} g(t))$ converges smoothly to a round metric.

The manifold doesn't have to be a sphere, but it has to be a three dimensional spherical space form. So these manifolds were all prime. If the Ricci curvature is strictly positive for a compact manifold then the fundamental group is finite by some argument involving [bounded geodesics?]

Okay, now we come to maybe the single most important ingredient in the whole story. I wish I had time to do it justice, but I don't. I'll have to leave it to you. This is the maximum principle.

We have the Ricci flow for the metric. The evolution equation for the scalar curvature is $\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2$. It's the easiest to write down. Take (M, g_0) with $|Rm(x,0)| \leq 1$. Then $-6 \leq R(x,0) \leq 6$. Then

Corollary 3 $R(x,t) \geq \frac{-6}{1+4t}$.

So how do you prove this? It's an example of the maximum principle. First of all, define

$$R_{\min}(t) = \min_{x \in M} R(x,t).$$

So first $R_{\min}(t)$ is continuous in t. It might not be smooth if you have more than one minimum.

[Is it piecewise smooth?]

You don't really know where these singularities are happening. But if $R(x,t) = R_{\min}(t)$, then $\frac{\partial R}{\partial t}(x,t) \geq \frac{2}{3}R^2$. This is because $2|Ric|^2 \geq \frac{2}{3}R^2$. At a minimum, $\Delta R = 0$ so you have the indicated inequality. So $\frac{d}{dt}R_{\min}(t) \geq \frac{2}{3}R_{\min}^2$ in terms of forward difference quotients.

So if you start with a lower bound you can use this bound to give a bound for all time. In the case $R \geq -6$ it is the indicated bound.

I want you to think about making this maximum principle tensorial. Let Z be closed in a tensor bundle V over (M, g) with

1. Z invariant under parallel translation.
2. $Z \cap V$ is a convex set in V_x. If $\frac{\partial \Phi}{\partial t} = \Delta \Phi + \phi(\Phi)$, is the evolution equation, where ϕ is a vector field tangent to the flow, and if ϕ preserves Z then any solution that starts in Z stays in Z.

5 Robert Lipshitz, Extending Heegard Floer Knot homology to higher genus boundary

My website is at math.stanford.edu/ lipshitz.

1. Cylindrical reformulation of Heegaard Floer
2. Heegard diagrams with boundary
3. Two-manifolds$\rightsquigarrow$ differential algebra A
4. 3-manifolds with boundary $\rightsquigarrow$ A-module CF.
5. Example
6. Confessions and future directions.

5.1

Let's let $(\Sigma_g, \underline{\alpha}, \underline{\beta}, z)$ be a pointed Heegaard diagram. Then $T_\alpha, T_\beta \hookrightarrow Sym^g(\Sigma)$. Then we get a chain complex $\widehat{CF} = \mathbb{F}_2\langle T_\alpha \cap T_\beta \rangle$, with differential d which counts holomorphic disks mapping to $Sym^g(\Sigma)$.

Okay, so $Sym^g(j)$-holomorphic ϕ are in correspondence with diagrams

$$\begin{tikzcd} S \arrow[r, "U_{\mathscr{E}}"] \arrow[d, "U_D"'] & \Sigma \\ D = [0,1] \times \mathbb{R} & \end{tikzcd}$$

with $U_D, U_{\mathscr{E}}$ both holomorphic. So count $(S, \delta S) \to \Sigma \times [0,1] \times \mathbb{R}, (\underline{\alpha} \times 1 \times \mathbb{R}) \cup (\underline{\beta} \times 0 \times)R))$. This is asymptotic to $\{x_i \times [0,1] | x_i \in \alpha_i \cap \beta_{\sigma(i)}$.

5.2

Let Σ_g be a surface with one boundary component. Inside σ let's have $\alpha_1, \ldots, \alpha_g$ circles, and then $\beta_1, \ldots, \beta_{2k}$ arcs with boundary on $\delta\Sigma$, and $\beta_{2k+1}, \ldots, \beta_{2k+g}$ circles in Σ. This collection $(\Sigma, \underline{\alpha}, \underline{\beta})$. This is a Heegaard diagram with boundary if, we can take two copies of Σ, unioned with opposite orientation across the boundary. Then we can take $(\Sigma \cup \bar{\Sigma}, \alpha \cup \bar{\alpha}, \beta \cup \bar{\beta}$. The original thing is a Heegaard diagram with boundary if the resulting object is a Heegaard diagram. Let me call the original thing H and the new thing $2H$. So $2H$ leads to a closed 3-manifold Z with an involution τ. Then we can define the manifold with boundary $Y(H)$ specified by H to be Z/τ, which we can denote $\frac{1}{2}Z$.

Exercise 9 *What is Y of the example I drew?*

Exercise 10 *What kinds of Morse functions give Heegaard diagrams with boundary.*

We should also fix $z \in \delta\Sigma$. So okay, we've done one, and we've nearly finished two.

Now we want to count holomorphic curves in such a thing. We can view $\delta\Sigma$ as a puncture which I'll call p. So now we have a surface with a long puncture instead, this very long neck. Now the β-arcs go out to the puncture. We're going to consider $\Sigma \times [0,1] \times \mathbb{R}$ again. The boundary is going, again, to $(\underline{\alpha} \times 1 \times \mathbb{R}) \cup (\underline{\beta} \times 0 \times \mathbb{R})$. Now there are three places we have to understand the asymptotics. There are $\pm\infty$, just the same as before. There is also another place I'll call east infinity. They are these g-tuples $\{x_i \times [0,1] | x_i \in \alpha_i \cap \beta_{\sigma(i)}\}$. Note that you don' tuse all the β_i. At east infinity, there is a $\gamma_i \times 0 \times t$, where this is an arc in $\delta\Sigma$ with $\delta\gamma_i \subset \underline{\beta} \cap \delta\Sigma$.

I have the point z in the boundary, and I want to avoid that point, as in $\widehat{HF}$. I don't know how to generalize that, it's part of part six.

That's the end of part two. The next part will essentially be independent of what I've said.

5.3 2-manifolds$\rightsquigarrow$ algebra $\mathscr{A}$

So let's say I have a circle with some marked points, I can identify the boundary and get a surface with genus one quarter the number of points. Then I label the arcs in the picture.

So I'm almost ready to tell you what $\mathscr{A}$ is.

Definition 3 *A* 2*-level ordered list of Reeb chords is a string of Reeb chords seperated by* $<$ *and* $<_\epsilon$ *. It's just a list of the* γ *arcs I drew in the picture sepereated by these symbols.*

$$\mathscr{A} = \mathbb{F}_2\langle\{2 \text{ level ordered lists of Reeb chords}\rangle\}.$$

Here multiplication is concatenation with $<$. So $(\gamma_1 < \gamma_2) \cdot (\gamma_4) = \gamma_1 < \gamma_2 < \gamma_4$.

I'll define two commuting differentials. One I'll call decol, where $\mathrm{decol}(O)$ is the sum over all ways of replacing $<_\epsilon$ with $<$.

$$\mathrm{decol}(\gamma_1 < \gamma_2 <_\epsilon \gamma_3 <_e psilon \gamma_4) = \gamma_1 < \gamma_2 < \gamma_3 <_e psilon \gamma_4 + \gamma_1 < \gamma_2 <_\epsilon \gamma_3 < \gamma_4$$

The other differential is the join. So $\mathrm{join}(O)$ is the sum over all ways of replacing $\gamma_i <_\epsilon \gamma_j$ by γ_{ji}. So for example

$$\mathrm{join}(\gamma_1 <_\epsilon \gamma_3 <_\epsilon \gamma_2 < \epsilon\gamma_1) = \gamma_1 < \epsilon\gamma_{23} <_\epsilon \gamma_1 + \gamma_1 <_\epsilon \gamma_3 <_\epsilon \gamma_{12}.$$

So d in the complex is $decol + join$.

[How does it compare to the standard differential?]

I think it's different. For example, we're not, let's talk about it later.

[In the definition of join, i and j are [unintelligible]?]

They're lists of numbers. It's bad notation.

Okay, so this one is really easy.

Exercise 11 *Check that* join^2 *and*

$$\mathrm{decol}^2 = \mathrm{join} \circ \mathrm{decol} + \mathrm{decol} \circ \mathrm{join} = 0$$

Exercise 12 $H_*(\mathscr{A}) = \mathbb{F}_2^4$*, where it's generated by* $\langle \emptyset, \gamma_1, \gamma_2, \gamma_3 \rangle$.

5.4

Okay, so the next thing we wanted to do was to associate to a 3-manifold with boundary a differential $\mathscr{A}$-module. Except for the differential this is very easy. Let $CF(\Sigma, \underline{\alpha}, \underline{\beta}) = \mathscr{A}\langle\{\text{internal points } x, \underline{x} = x_i \in \alpha_i \cap \beta_{\sigma(i)}\}$.

The differential is of course counting holomorphic curves. So given $O, \underline{x}, \underline{y}$, let $\mathscr{M}^0(\underline{x}, \underline{y})$ be the set of holomorphic maps $S \to \Sigma \times [0,1] \times \mathbb{R}$ which are asymptotic to $\underline{x}$ at $-\infty$, to $\underline{y}$ at $+\infty$, and 0 at ∞.

[How do you deal with $<_\epsilon$?]

Really I look for something where you have a splitting like this. You'd like it to split it apart, and it's a combinatorial issue. I could have said at the beginning, at my webpage is work in progress on this that I would be happy to discuss with people.

Okay, I'm getting to the punchline. $d\underline{x} = \sum_y (\#\mathscr{M}^0(\underline{x}, \underline{y}))O \otimes$. Then extend by Leibniz.

Theorem 15 $d^2 = 0$.

Theorem 16 *The homotopy type as a module over $\mathscr{A}$ of CF is an invariant of the bordered manifold $(Y, \delta Y)$.*

This is an invariant of manifolds with parameterized boundary, just like Floer homology.

[[unintelligible]?]

Yes, that's true, in small genus it corresponds to [unintelligible].

5.5

Let me give you an example.

Here's a torus, here are α and β, and z. Okay, so $dr/,\ ds = r + \gamma_1(t) + (\gamma_2, < \epsilon\gamma_3)r + \gamma_{23}r, dt = \gamma_3 r$.

You can check directly that $d^2 = 0$. This is a Heegaard diagram for a solid torus. Let me draw one more. [Pictures.]

So what I won't have time to do is show that there are maps of three pictures of the solid to one another which fit into a short exact sequence, like the triangle in Heegaard Floer homology.

5.6

I faked it slightly with $\mathscr{A}$, it's got a little more info than I let on.

There's also, I don't know what happens when you have two manifolds with the same boundary. I also don't know what happens when [unintelligible]. In the case of torus boundary this is related to knot Floer homology. So Matt and [unintelligible]have all this wonderful work with satellite knots. It would be nice to understand knot or tangle invariants in terms of it, and how it relates to genus embedded surfaces.

[There's this invariant of sutured manifolds. [unintelligible]is there any relationship there?]

I haven't read that work yet?

[Can you improve this to $\mathbb{Z}$ coefficients?]

I assume the answer is yes, but keeping track of signs is the most unpleasant thing in the world. I hope when this is done, someone does it. I hope it's not me.

[Is there a nice class of examples where you can compute things combinatorially?]

I assume the answer is yes. It's certainly nice for some class of knots but it might be the unknot. Not so clear.

[[unintelligible]]

Matt's saying that you can get the pattern for a lot of satellite knots from a genus one Heegaard diagram. But you need the gluing theorem. Hopefully it won't take that long. There may be a seperate proof for the genus one case.

[[unintelligible]]

There's a huge grading, like $\mathbb{Z}^{4g-1}$ or something. That passes to a filtration on the chain complex.

[What about the relationship with the doubled manifold? Is that a baby case of the surgery?]

It looks like I should have an answer for that but I don't. Here's my excuse. The complex structure gets conjugated in the reflection. I take it back, I don't have a good excuse. Matt points out that if it were a knot complement, it's determined by knot Floer homology.

[Is this determined by knot Floer homology?]

I don't think so, but maybe.

[Let's thank the speaker again.]