Lusztig Conference Notes
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Gabriel C. Drummond-Cole

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1 Roman [unintelligible], MIT, a geometric approach to modular representations of semisimple Lie algebras

Let me start with some standard notations. We'll work over an algebraically closed field k and then we'll have G and $\mathfrak{g} \supset \mathscr{N}$ the nilpotent cone, and then we have $\mathscr{B} = G/B$ and $T^*\mathscr{B} = \tilde{N} \to N$ which can be written $\{b, x | x \in rad\ b\} \mapsto x$ which is also the moment map for the standard symplectic form.

I should say, as source,

1. This is a realization of the affine Hecke algebra of the Langland dual group $G^\vee$. This is some algebra over the ring of Laurent polynomials. In some form this goes back to a paper of [unintelligible], it can be realized as [unintelligible]and the product is by convolutions. You can realize the standard module over the affine Hecke algebra. It contains the finite Hecke algebra, and so you get the antisymmetrizer.
2. This is some conjectural description of a canonical basis in the cohomology of the Springer fiber $H^*(\mathscr{B}_e)$ and a relation to modular representations of $\mathfrak{g}$.

What I am doing is

I. Categorification of 1, well, using the Grothiendieck-[unintelligible]function philosophy, it is a K-group over some other.

II. If you like it's a kind of categorification of 2, geometric methods for modular representations of $\mathfrak{g}$.

You can apply I to II to show canonicity of the basis.

III. A generalization of methods of II. to other geometric contexts, namely the cotangent bundle of a flag variety is replaced by some other algebraic variety. One can replace it with any resolution of singularities carrying a symplectic form. I'll discuss it carrying the Hilbert scheme of points on a plane. Then it is the modular representations of $\mathfrak{g}$ are replaced by modular representations of rational [unintelligible].

Let me talk briefly about I. If you have $\mathscr{H}_{aff} \otimes \mathbb{Q}$ with $q \mapsto p^a$ then this is $\cong \mathbb{Q}[I \backslash G^\vee(F)/I]$ where $F \cong \mathbb{F}_p((t))$ and I is the Iwahori group. The quotient $G^\vee(F)/I \cong \mathscr{F}(\mathbb{F}_{p^a})$ where $\mathscr{F}$ means the affine flag variety.

So $\mathscr{H}_{aff} \iff D_I(\mathscr{F}$.

We have $M_{\text{asp}} \cong \mathbb{C}[I \backslash G^\vee(F)/N(F), \psi]$ where N is the maximal nilponents and ψ is a non-degenerate character.

Theorem 1 *$D^b(Coh^G(\tilde{N})) \cong D^b(P_{\text{asp}})$ where this is antispherical perverse sheaves on $\mathscr{F}$.*

So I will now pass to II.

Let *char* $k = p > h, U = U(\mathfrak{g})$. So $Z = Z(U) \supset U^G$ but also the Frobenius center $\langle X^p = X^{[p]}\rangle$.

So a central character is specified by (λ, e). It's known from old work that it's enough ot study $e \in \mathscr{N}$ and then $\lambda \in {}^*_{\mathbb{F}_p} = \Lambda/p\Lambda$, where Λ is the weight lattice.

So U_e^λ sits inside U with an intermediate thing $U_{\hat{e}}^\lambda$ in between for technical reasons. So say λ is regular if its stabilizer is trivial. Then

Theorem 2 *–, Markovic, [unintelligible]*
If λ is regular, then the bounded derived category $D^b(U_{\hat{e}}^\lambda - mod^{f.g.})$ is a full subcategory of $D^b(coh(\tilde{N}))$.

There are variations of this.

Corollary 1 *the number of irreducible representations of U_e^λ is the same as the dimension of $H^*(\mathscr{B}_e)$.*

Some of the more difficult elements of this construction arise as follows, from the compatibility of I and II: The irreducible objects in the derived category of coherent sheaves on the cotangent bundle, which corresponds to irreductible representations of U_e^λ, as follows: for some $F \in P_{asp}$ such that, let $\mathscr{C}_F$ be the corresponding sheaf, then we can find [unintelligible]so that $\mathscr{C}_F$ is supported [unintelligible]. Then the irreducible representation is a direct summand in $\mathscr{C}_F$ restricted to π^{-1} of a slice to $G(e)$, up to a shift.

Now I want to just explain a little bit about how this works. So in characteristic zero and for a regular dominant λ an equivalence between U^λ-modules and a category of D-modules on the flag variety.

At this point I ceased taking notes.

2 Beilinson

It's a pleasure to be back here. I would like to thank George for so much great mathematics.

Recall that ϵ-factors have two sides. They come as constants in equations for Galois type functions. Let V be a finite dimensional graded vector space along with a self-map which is frobenius. Then $L(V,t)$ is $\prod det(1 - Fr\ t, V^i)^{(-1)^{|i|}}$.

Then $\epsilon(V,t) = \prod det(-Fr\ t, V^i)^{(-1)^{|i|}}$.

I couldn't see the board.

3

It's a great pleasure for me to speak here. Today I want to speak on graded lift. I will let $\mathscr{O}$ be $\{M$ in the $\mathfrak{g}$-modules which are finitely generated and locally finite over $\mathfrak{b}$ and semisimple over $\mathfrak{h}\}$. So let $\mathscr{F}$ be $\mathfrak{g}$-modules, finite dimensional, with $\otimes$. Then $E\otimes : \mathscr{O} \to \mathscr{O}$. Then $\mathscr{O}$ admits a $\mathbb{Z}$-graded version $\mathscr{O}^{\mathbb{Z}}$. Then does $E\tilde{\otimes}$ exist as a functor that lifts $E\otimes$.

The first question is whether every functor $E\otimes$ admit a graded lift. The answer is yes. Lots and lots of them. I will discuss this later. The second question is, can we choose these graded lifts such that $(E_1 \oplus E_2)\tilde{otimes} = (E\tilde{\otimes}) \oplus (E_2\tilde{\otimes})$? How about that $(E_1 \otimes E_2)\tilde{\otimes} = (E_1\tilde{\otimes}) \circ (E_2\tilde{\otimes})$. The answer is also yes but it is not so easy.

The third question is whether we can lift the action of $(\mathscr{F}, \otimes)$ on $\mathscr{O}$ to an action on $\mathscr{O}^{\mathbb{Z}}$? The answer is no way. The isomorphisms are not canonical at all, and cannot be made canonical. Between the second and third question there is much work to be done, I haven't done that.

The first thing is about the graded version of this category $\mathscr{O}$. This is old work, based somehow on ideas of [unintelligible]. This category $\mathscr{O}$ breaks up as $\oplus_\lambda \mathscr{O}_\lambda$. This $\lambda \in \mathfrak{h}^*_{dom}$. So $\mathscr{O}_\lambda \cong A_\lambda$-modules. Now A_λ admits a $\mathbb{Z}$-grading with only positive degree parts and A^0_λ semisimple. We insist that A^1_λ generate A_λ over A^0_λ. This is not unique but the ring you get is.

So wipe this out and you get A_λ-graded modules over this algebra. Then you have the forgetful functor v and so you think that $\mathscr{O}^{\mathbb{Z}}_\lambda$ is this category and to get the whole category you take the sum of the blocks $\oplus \mathscr{O}^{\mathbb{Z}}_\lambda$.

The graded objects are in correspondence with $\mathfrak{h}^* \times \mathbb{Z}$ by $(\mu, n) \mapsto L^{\mathbb{Z}}(\mu)\langle n\rangle$.

To convince you that this graded category is notural to consider, although the Verma module admits a graded lift $\Delta^{\mathbb{Z}}(\mu) \twoheadrightarrow L^{\mathbb{Z}}(\mu)$.

Then $[\Delta^{\mathbb{Z}}(x \cdot 0) : L^{\mathbb{Z}}(y \cdot 0)\langle i\rangle]$ is some coefficient of a KL-polynomial.

There's a last thing I want to say, that is, a trivial example. Take $\mathfrak{sl}_2$. Then $\mathscr{O}_0$ is equivalent to the quiver $\bullet \rightleftarrows \bullet$ with the condition that going once around starting at the left vanishes.

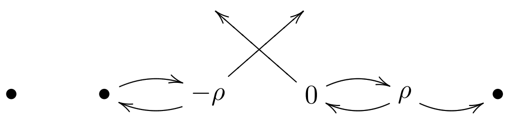
In $\mathscr{O}_0$ it is the same thing except each arrow has degree one and representations are in graded vector spaces.

If I have a functor $\mathscr{O}_\lambda \to \mathscr{O}_\mu$ which is a projective functor, a direct summand of a functor $E \otimes$. Considering these up to isomorphism, they correspond with projective objects in $\mathscr{O}_\mu$. To each functor I associate its evaluation on the Verma module.

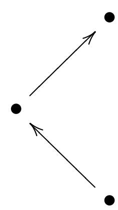
Now, what should I say, somehow it's not difficult to prove that they lift and somehow, all these functors admit lifts and the possible lifts are parametrized by gradings on a projective module. It's a big mess if you have more than one summand.

Let me perhaps explain once more how to do this in $\mathfrak{sl}_2$. Say, this functor I just erased, this representation tells you that a graded lift is well-determined by its action on the Grothiendieck group. I take the two-dimensional representation and I want, I want to grode the tensor with every finite dimensional represtentation.

So $L(\rho)\tilde{\otimes}[\Delta^{\mathbb{Z}}(i)]$ is $\Delta^{\mathbb{Z}}(i+1) + \Delta^{\mathbb{Z}}(i-1)$ for $i \neq 0, 1$, and then $\Delta(-1)\langle 1\rangle + \Delta(1)$ for $i = 0$ and $\Delta(-2) + \Delta(0)\langle 1\rangle$ for $i = 1$. So this looks like



So we get in the antidominant projective:



Then you get

			1	
0		1		
$-\rho$	0	$Z, 0$		
-2ρ		0		
			0	

These ad hoc [unintelligible]don't really help in general.

If you take just integral weights, this is the same size as $\tilde{H}\underline{H}_{\omega_0}$. We can take the shortest. This lambda, this is shortest in $e^\lambda v$. This is in the affine Weyl group.

A Verma module $\Delta(\lambda - rho\langle i\rangle \mapsto v^i V_\lambda$. This gives, by $v = \sqrt{q}$, [unintelligible].

Then $E\tilde{\otimes}$ corresponds to a Hecke algebra element. I want it to be spherical. Then [unintelligible]. We have $\langle i\rangle \iff \cdot v^i$.

Let me describe how to find this thing, basically there is just some positivity to check. Namely, I claim the following statement. Which way should I put it? It is pretty clear what is the underlying picture, I should take the affine Grassmannian, I should think of it as G/P. Really it's the loop group divided out by the disk group $G((t))/P[[t]]$. It's the geometry corresponding to this model. This acts basically, take P-equivalent objects on G/P. This will act on this thing here. So we have this Grassmannian, and now if we take sheaves on this Grassmanian, take $Der_I(Gr)$ then $\times Der_P(Gr)$ then I'll have a convolution, giving a wholly equivariant sheaf on the Grassmanian, $Der_I(Gr)$. The point is the following. This is somehow a theorem. The essential theorem in this whole story

Theorem 3 *Take the IC complex of Iwahori orbit, extend this, and then convolute it with an IC complex in this last category. Then restrict it with i^*. We have $PyP/P \hookrightarrow^i Gr \hookleftarrow^j PxP/P$. This all is perverse semisimple.*

An E-equivariant sheaf, semisimple on a parabolic thing in the flag variety, extend it by zero and then convolve it and restrict it to another parobolic orbit and it is semisimple again. I think this is a very funny property, and in fact, I should say somehow this semisimplicity gives me the combinatorial positivity needed to define these equivalences. It gives me much more but I did not prove it. This thing is an affine fibration of a parabolic flag manifold. This is like one block of category $\mathscr{O}$, well, is Kozsul dual to it. If I fit anybody in here, it's Kozsul dual to doing [unintelligible]. This is a good point to stop.