[Nadler: What about singularities?]

[You can definitely formulate things, but I don't know the best thing that's true. The application I'll describe today needs singularities.]

The previous lecture was motivation for what I'll assert in this lecture.

[If you go back to a $K(A, n)$, what is a description for that version of homology? hocolim$Map_c(U, X)$, where U are disjoint unions of disks and X is $K(A, n)$. I could write it as $X = \Omega^\infty E$; then I could do the same thing with spectra with maps into E, and I could look instead at Ω^∞hocolim$Map_c(U, E)$, and if things are connected enough then this is a homotopy equivalence. In spectra, the coproduct and product are the same, and you can thus use a single disk. Then this is the homology of M with coefficients in a local system.]

All right, so today I'd like to bring the last two lectures together and apply this noncommutative Poincaré duality. This is unfinished joint work with Dennis Gatesbury [sp?] I'm going to give some notation now at the beginning.

$\mathbb{F}_q$ is a finite field with q elements. X is a smooth complete algebraic curve over $\mathbb{F}_q$. K is the fraction field of X. If X were over $\mathbb{C}$, this would be a Riemann surface, and this would be meromorphic functions on this Riemann surface.

$x \in X$ is a closed point, and $\mathcal{O}_x$ is the complete local ring of X at x, functions defined on a small disk around x, and K_x is a fraction field of $\mathcal{O}_x$. The ring of adeles $\mathcal{A}$ is the product (restricted $\prod_x^{res} K_x$

Now G_0 will denote a semisimple, simply connected algebraic group over K, and as a consequence of being an algebraic group over K, we can evaluate G_0 on a ring containing K. So in particular, you can talk about $G_0(\mathcal{A})$. The adeles, I should say, are locally compact, so $G_0(A)$ is a locally compact group. It has a canonical Haar measure as in the first lecture, which I'll denote μ_{Tam} for Tamagawa. This contains $G_0(K)$ diagonally embedded, which is discrete. Given all this notation, I can formulate what we are trying to prove, namely Weil's conjecture, that

$$\mu_{Tam}(G_0(K)\backslash G_0(\mathbb{A})) = 1$$

We started with the classification of quadratic forms in the same genus as a given form. When we phrased it in terms of the measure of a coset space. Now I would like to undo the steps of the first lecture and rephrase this as a counting problem.

The group corresponding to our count in the first lecture was SO, which is the automorphism group of the quadratic form. So what sort of objects will we count? This group G will be the automorphisms of something.

I'll make a choice. Inside the algebraic curve X there is its generic point $Spec\ K$, and G_0 starts its life as an algebraic group over K. Let G be a group scheme over X with generic fiber G_0. Let me add the adjective "nice," which I'll use in a non-technical sense.

Remember, we can evaluate G_0 on any K-algebra. So $K \subset K_x$, so it sits in the adeles, but K is not in $\mathcal{O}_x$. So a priori we can't talk about $G_0(\mathcal{O}_x)$. Now we have an integral structure now, so we can talk about $G(\mathcal{O}_x)$. We could talk about quadratic forms over any ring, starting from the integers.

Our coset space in the first lecture was $SO_q(\mathbb{Q})\backslash SO_q(\mathbb{A})$. For this we only needed a rational version, but to go back to the integers, we needed to mod out on the other side $SO_q(\hat{\mathbb{Z}}) \times SO_q(\mathbb{R})$. For that we need integrality.

Let me say something about “nice.” [missed some.] If I want to get as close as I can get to semisimple using general existence theorems.

I wrote G_0 before but now we will write $G(K)\backslash G(\mathbb{A})$ and now we can mod out on the other side by $\prod G(\mathcal{O}_x)$. This, just like in the first lecture, has a combinatorial description. It's the set of isomorphism classes of principal G-bundles on X. The identification is not entirely trivial, and relies on the Hasse principle. Any G-bundle is trivial over a generic point of the curve, which requires the curve to be, say, simply connected.

How do we connect the double coset to this? $G(K)\backslash G(A)$ inherits a Haar measure, and is acted on by $\prod G(\mathcal{O}_x)$. If the action is free, this should be the number of double cosets times the measure of $\prod G(\mathcal{O}_x)$. These are compact so this would be finite.

This isn't quite correct because the group doesn't act freely. What is correct is that $\mu(G(K)\backslash G(\mathbb{A}))$ is $\mu(\prod G(\mathcal{O}_x))$ times a mass term, which is

$$\sum_P \frac{1}{|Aut\ P|}$$

where P varies over G-bundles. The automorphisms over a finite field are finite. This sum is infinite, and I'm claiming it converges to something that makes this true.

This is the kind of statement that we want to prove. You should find that $\sum_P \frac{1}{|Aut\ P|}$ is $\frac{1}{\mu \prod G(\mathcal{O}_x)}$ which is $q^d \prod \lambda_x$. (this is by definition).

Let me scale back my goals. I'm not going to get to details about what the λs are. I'll show that the left hand side has a presentation of this form. I'll leave out the problem of showing they are the same local factors as in Tamagawa measure.

That's my goal now, to convince you that there is a formula of this type.

Informally, let me call the left hand side “the number of G-bundles on X.” This is what we want to do, we want to count the G-bundles on X. So far this is parallel to the story in the first lectures. We had the number of quadratic forms in the same genus, which was a set. This has a structure, because G-bundles on X are parameterized in an algebro-geometric way.

$Bun_G(X)$ is the moduli stack of G-bundles on X. That means $Bun_G(X)$ can be evaluated on R containing $\mathbb{F}_q$, where these are G-bundles on $X \times Spec\ R$ (over $Spec\ \mathbb{F}_q$).

This is like a variety, but we want to think of it as a groupoid whose objects are G-bundles and whose morphisms are automorphisms.

Then what are we doing? What are we interested in? I can rewrite the left-hand side, we want to compute the number of points in $Bun_G(X)(\mathbb{F}_q)$. When I say number, again, I mean counted with multiplicity.

Let's ignore that for a moment and say that Y, as a warmup, is an algebraic variety over $\mathbb{F}_q$ and we ask a question like “what is the number of points in Y defined over $\mathbb{F}_q$. This is the subject of other famous conjectures of Weil. There is a map called the Frobenius map $Frob : Y \to Y$. You take all the coordinates and raise them to the qth power. The equations definining Y are preserved. If you want to know if Y is defined over $\mathbb{F}_q$, because this is the set of fixed points of the Frobenius map, that you should be able to count with the Lefschetz fixed point formula.

This should be a sum, an alternating sum of the traces

$$\sum_i (-1)^i Trace(Frob(H_c^i(Y)))$$

Let me call this the Grothiendieck-Lefschetz trace formula. The difficulty is making sense of $H_c^i(Y)$. Y is defined over a finite field $\mathbb{F}_q$, but Grothiendieck introducted étale cohomology, and then you can make sense of the right hand side and connect it to the left-hand side.

Let me specialize to Y being smooth of dimension d. Then you have Poincaré duality. Frobenius is not compactible with it, and you learn that $Y(\mathbb{F}_q)$ is given by $q^d \sum_i (-1)^i Tr(Frob(H_i(Y)))$, which I can phrase as

$$\frac{\#Y(\mathbb{F}_q)}{q^d} = \sum (-1)^i Tr(Frob(H_i(Y)))$$

Let me now apply this when Y is not an algebraic variety but instead $Bun_G(X)$. The points of $Bun_G(X)$ over $\mathbb{F}_q$ over q^d will be a sum $\sum (-1)^i Tr(Frob|H_i(Bun_G(X)))$

[missed some]

Let's remember our goal, to prove that this expression is a product of local factors λ_x.

What are we doing now? Now it's a topological problem, to find the alternating sum of the trace of Frobenius on the homology of the moduli stack.

This is all standard. Let me input the new idea, which is to use non-Abelian Poincaré duality. Let's imagine that G came from an algebraic curve defined over $\mathbb{F}_q$. This is a map from X into BG. We're asking about the homology groups of the mapping space $Map(X, BG)$. That's non-Abelian cohomology and you can rewrite it to unintelligible.

We saw that we could build this as a homotopy colimit. Now let's translate to algebraic geometry. We can take last lecture as motivation, and write down some algebro-geometric objects relevant to this description. Let me define $Ran_G(X)$, which will parametrize the following: you should give a finite set S, a map $U : S \to X$, a bundle P on X, and a trivialization away from $U(S)$. These are G bundles concentrated at these special points. $Ran(X)$ is the same thing when G is trivial, finite sets $S \to X$. $Ran_G(X)$ maps into $Bun_G(X)$ by forgetting verything but the bundle and into $Ran(X)$ by forgetting everything but the finite sets.

The map to $Bun_G(X)$ is a homology equivalence (probably a homotopy equivalence). Now we can rephrase again. Our goal now is to compute the alternating sum $\sum (-1)^i Tr(Frob|H_i(Ran_G(X))$.

So let's do this by analyzing the map ψ to $Ran(X)$. We need sheaf theory for this calculation. This means compactly supported cohomology with coefficients in the dualizing sheaf. Let me write K_{Ran_G} for the dualizing sheaf. Se we can rewrite this as

$$\sum (-1)^i Tr(Frob|H_c^i(Ran_G, K_{Ran_G})$$

We can take the compactly supported image under ψ and then maeasure what's left.