Today I'd like to talk about something that is a priori really different, something I'll call non-Abelian Poincaré duality. Let's review the usual Poincaré duality. LEt M be a compact oriented manifold of dimension n (without boundary). The cohomology $H^*(M, A) \cong H_{n-*}(M, A)$. To get this isomorphism we need a few assumptions. The most important is that M is a manifold, which is a local condition, that every point has a neighborhood looking like Euclidean space.

You can't do this for Euclidean space itself, unless the dimension is 0. We can change this to use non-compact manifolds if we use H_c^*, compactly supported cohomology.

Let's state the proof of Poincaré duality in two parts. Euclidean space is contractible, and so you should just get A in degree 0. Then the stament is that the compactly supported cohomology with coefficients in A is 0 except in top degree, where it is n. Now we'd like to understand why this is true for an arbitrary manifold. Ultimately I'd like to talk about cohomology and homology groups. For now let's talk about chain complexes. If U is a subset of M, let $C_*(U, A)$ and $C_c^*(U, a)$ be the singular chain complex or compactly supported cochain complex of U, and from now on I'll omit the letter A. These are chain complexes, covariant functors of U, if you consider $U \to C_*(U)$ or $U \to C_c^*(U)$, you can extend compactly supported cochains by 0. Poincaré duality unintelligibleboth of these are (homotopy) cosheaves of chain complexes.

To be a sheaf, to every open set you have an Abelian group, and there is a gluing condition when you have two open sets and their intersection. You can talk in this covariant situation about a dual condition. If $U, V \subset M$, we can look at the diagram

$$\begin{tikzcd} C_*(U \cap V) \arrow[r] \arrow[d] & C_*(U) \arrow[d] \\ C_*(V) \arrow[r] & C_*(U \cup V) \end{tikzcd}$$

You could ask that this be a pushout, which is unreasonable, but it's a homotopy pushout, so you have a quasiisomorphism, which gives you a Mayer-Vietoris long exact sequence

$$H_i(U \cap V) \to H_i(U) \oplus H_i(V) \to H_i(U \cup V) \to H_{i-1}(U \cap V) \to \cdots$$

This isn't quite the condition for a cosheaf, you'd want to say something similar for arbitrary collections of subsets. These are homotopy cosheaves of chain complexes. Both of them satisfy some sort of excision, have Mayer-Vietoris sequences. If you have a sheaf on a topological space, it's described by what happens on small open sets. In this context, the same thing is true. M is a manifold, so it has a basis of open sets looking like Euclidean space.

A consequence of being a homotopy cosheaf, if I'm interested in $C_*(M)$, I can take the colimit over $U \subset M$ of $C_*(U)$, which maps to $C_*(M)$ by functoriality. This map, done correctly, will be a quasiisomorphism. Thus you can know what your chain complex looks like by understanding it on an appropriate basis. The same statement holds on compactly supported cohomology for the same reason.

Now, I claim we are done. The local calculation tells you how to construct the isomorphism of the chains. We can write down a term-wise quasiisomorphism. The local isomorphism depends on an isomorphism with Euclidean space, and if you want to choose these compatibly, then you need the manifold M to be orientable. Then you can write down the desired quasiisomorphism and it passes to homology.

This is the kind of thing I want to bring to the non-Abelian side. What if we want to talk about non-Abelian cohomology. Let me start by saying what non-Abelian cohomology is.

Cohomology, recall, is a representable functor on the homotopy category of spaces. If you have a nice space M and you are interested in $H^n(M, A)$, this is the same as $[M, K(A, n)]$, where $K(A, n)$ is a space whose homotopy groups vanish except $\pi_n(K(A, n)) = A$. There is such a space, unique up to homotopy equivalence.

What is non-Abelian cohomology about? For example, when $n = 1$, we can understand $K(A, 1)$ if A is not Abelian. BG has fundamental group G and no higher homotopy groups. Then $H^1(M, G)$ could be taken to be $[M, BG]$, which is the set of isomorphism classes of G-torsors on M. If you can assume that M is connected with a chosen basepoint, this is the set of conjugacy classes of maps from $\pi_1(M)$ to G.

This is one notion of non-Abelian cohomology, studying maps into BG. If you take this, it suggests a generalization. Let's drop the assumption we're mapping into an Eilenberg-MacLane space and map into any space whatsoever. For any space X, we could define $H(M, X)$ to be $[M, X]$. When $X = K(A, n)$, this recovers $H^n(M, A)$. If $X = BG$ you recover the definition just stated.

The question this lecture is addressed to, if this is like cohomology and M is a manifold, this should also be some kind of homology. What is the analog of Poincaré duality in this setting.

Let me make sort of a dictionary of how these ideas will match up in the Abelian and non-Abelian cases.

Abelian
an Abelian group A and degree n
$H^n(M, A)$ is a set with a group structur
$H^n_c(M, A)$
$C^*(M, A)$
$C^*_c(M, A)$

I wanted to talk earlier about compactly supported cohomology.

This is the analogy we are following.

One side of Poincaré duality will tell us about compactly supported cohomology. So we want to talk about $Map_c(M, X)$ "via homology."

The proof was in two steps, locally and then by going from local to global. Let's first consider the local case, where $M = \mathbb{R}^n$ (a different n than before).

What is $Map(\mathbb{R}^n, X)$? This means you're supported in a ball of radius r, which means I don't care what r is, so I might as well shrink it to radius 1. So then that is a ball where the boundary is taken to basepoint, which is the n-fold loop space of X.

I can look at compactly supported maps from U into X. We can ask the question, is the functor that takes U to $Map_c(U, X)$ a homotopy cosheaf on M? I should probably say what the target category is. That should take open sets to spaces. If it is, we can recover the global sections by a colimit over a convenient basis. The answer is that it can't be right.

What would this be saying? We could draw our diagram

$$\begin{tikzcd} Map_c(U\cap V,X) \arrow[r] \arrow[d] & Map_c(U,X) \arrow[d] \\ Map_c(V,X) \arrow[r] & Map_c(U\cup V,X) \end{tikzcd}$$

Then we could find every compactly supported map either in U or in V, which is plainly impossible (check the circle).

This is a different category. Earlier we were taking a homotopy cosheaf, you had a Mayer-Vietoris thing. We could pull something back to a sum of things from U and V. We're in the non-Abelian setting, so we can't add.

There is a situation where it makes sense to combine two such maps. If the sets are disjoint, then we can look at compactly supported maps from the union, which will be the product, $Map_c(U \cup V, X) \leftarrow Map_c(U,X) \times Map_c(V,X)$. This is like addition in chain complexes.

The idea now is to make use of this. In other words, this is not a homotopy cosheaf of spaces, but it's not just any functor, it's a functor with this sort of special feature. If we modify our notion a little bit, maybe we can salvage our idea. What do we want to say? Pretend for a moment the answer had been yes, what would we have done then? Then we would have proceeded as in the Abelian case, looking at $Map_c(M,X)$, and that receives a map from the colimit of $Map_c(U,X)$ over disks U, which is the n-fold loop space of X. Then this would be a homotopy equivalence. This won't be a homotopy equivalence because we haven't used the factorization property. We've only used things that look like disks, and then you don't get closure under disjoint union. Now let's take the colimit when you have a disjoint union of finitely many disks.

Theorem 1. *If X is $n-1$-connected, then this map from the colimit to $Map_c(M,X)$ is a homotopy equivalence.*

I want to say that the left hand side is some kind of homology, because it's a colimit, but maybe not of M but of something related.

The hypothesis is needed. Suppose M is a circle, and X consists of two points. One is the basepoint.

Every map from M to X is compactly supported. There are then two maps. $Map_c(M,X)$ has two points. But on the left hand side, a map on the other side is not homotopic to anything supported on a proper subset of S^1. The connectivity is needed. You could generate a similar example in higher dimensions, and I'll leave that to you.

Now let me sketch a plausibility argument for why you might think this is sufficient.

On the left, you have something that's a homotopy colimit over a big category. In degree zero you can describe this, π_0 of this space, we can ask about this being surjective on π_0. This means is that any compactly supported map f from M into X is homotopic to a map which is supported in a finite union of disks. Once we've got a map in a finite union of disks, then we can get it small. Then this is concentrated near a finite set. The global statement should be thought of as saying that can be done with parameters, canonically.

Let's say that M is a smooth manifold, and then I can triangulate it. Let me assume for simplicity that it's compact. Look at the $n-1$-skeleton of M, and I can restrict f to the $n-1$-skeleton. The target space in $n-1$-connected. This map is nullhomotopic on the skeleton, and modify the map by a homotopy so it carries a neighborhood of the $n-1$-skeleton to the basepoint. Then the map is supported on the interior of the n-simplices, and so this is a map from a disjoint union of $n-1$-disks.

That argument should make the statement plausible. I don't know how to turn this ingredient into a proof of the statement.

Maybe I should mention an example that might be faimiliar to an algebraic topologist in the audience. Take M to be the circle, and X to be connected. Then what are we doing? The left hand side is a homotopy colimit over maps from U that look like finitely many intervals. So a disjoint union of intervals is just finitely many products, so of $(\Omega X)^k$ to the maps $S^1 \to X$, which are the free loops LX. This space on the left has a multiplication, coherent up to homotopy. It's homotopy equivalent to a topological group. You're taking powers of this group, and the maps are given by multiplication on the group, and this is the cyclic bar construction of the space X. In this case, the based loop space ΩX, the cyclic bar construction gives you the free loop space of X. This is more usually stated at the level of homology. Passing to homology groups, you have $H_*(LX, \mathbb{Q})$ on the right, and on the right you get the Hochschild homology of a certain differential graded algebra, namely $C_*(\Omega X)$.

I think what I'd like to do with the rest of the lecture is state more strongly that the left hand side should be thought of as homology (of the Ran space of M). Let me inttroduce, $Ran(M)$, I will assume that M is connected. This is the collection of nonempty finite sets $S \subset M$. If you chose a metric on M, you could say it's the maximum of the distance unintelligible, or you could write a basis, if $U_1, \ldots, U_k$ are disjoint open stes in M, then $Ran(U_1, \ldots, U_k)$ will be the collection S so that S intersects each U_i and is contained in the union of the U_i.

Here are some subsets, and I'll declare them to be open. This is a basis. That's the same as the metric topology I described a moment ago.

Now, let U be the collection of basic open sets in $Ran(M)$. Let U_0 be the subset of sets of the form $U_1, \ldots, U_k$ where each one is a disk.