

Berkeley Quantum Workshop

Gabriel C. Drummond-Cole

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1 Ingo Runkel

Yesterday we talked about relating quantum field theory and physics. We talked about the lattice model, looking at it when the points are separated by large distances. Now I want to connect the correlators to Segal's way of looking at CFTs. I will not necessarily write out all definitions and details. But all definitions are found in various places. A CFT is a continuous projective symmetric monoidal functor between Riemann surfaces with parameterized boundaries (composition is gluing) to topological vector spaces, $C : RS \rightarrow TV$. Projective means that composition isn't preserved on the nose but up to a scalar anomaly. We want that if two Riemann surfaces are close to each other then also the correlators will be close to each other.

It's difficult to write down examples that aren't also topological field theories. That means it's hard to make a mathematical statement about this structure. So I want to make a different definition. Let me say what makes this difficult. Here's a list of the things we want and the difficult things we want

	$C(S) = H$ a topological vector space	$F = \bigoplus F_{a,b}$, a sum of finite dimensions
things we can get:	C of a pair of pants should be a map $H \otimes H \rightarrow H$	
	Higher genus surfaces should correspond to operations	

So we will modify this. The first thing we want is to restrict to genus zero, and then replace the boundaries with punctures with local coordinates.

What I'm presenting now I took from a book by Huang and a paper by Huang and Kong. So we will formulate "genus 0 correlators" as a functor, which will not be an honest functor but a smooth "partial" projective symmetric monoidal functor which now goes $C : ST \rightarrow GV$, where ST is spheres with tubes. ST is really an operad, but to make the relation to RS I will treat it like a boundary. The objects are natural numbers. The morphisms from (m, n) will be conformal equivalence classes of disjoint unions of spheres with punctures with local coordinates. Each sphere can carry an arbitrary number of ingoing punctures and exactly one outgoing puncture. The punctures are $p_\alpha, U_\alpha, \varphi_\alpha$ where p_α is a point in the sphere S_i

contained in U_α a neighborhood, and $\varphi_\alpha : U_\alpha \hookrightarrow \mathbb{C}$ a holomorphic chart taking p_α to 0. Then each has one outgoing puncture (q_i, V_i, ψ_i) .

Now I will define composition. This is by cutting out little disks around the punctures and gluing the outgoing puncture to an ingoing puncture.

Suppose I have one sphere here and I want to glue to it another sphere. I take my map φ to a neighborhood in the complex plane, and I can do the same on ψ . I demand that I can find a disk in these two of radii r and $\frac{1}{r}$, and I can cut out the preimages of these disks, and I identify them using the map $z \rightarrow \frac{1}{z}$. This turns out to not depend (up to conformal equivalence class) on r . This is not always defined; I should also say that B_r and $B_{1/r}$ should be contained in U_α and V_i and we should demand that the preimages $\varphi^{-1}(B_r)$ and $\psi^{-1}(B_{1/r})$ do not contain any punctures. So I cannot always glue.

That is why the composition is not always defined. Maybe I will make this comment. This structure, rather than saying I want a category with partially defined composition, I can say that it is a partial operad. There is a full subcategory where this is fully defined, where these U_α and V_i must contain in their image the ball of radius one.

Let's look at one example. I want to look at a morphism that sits in $A_\lambda \in \text{Hom}(1, 1)$, A for annulus. I will take $A_\lambda = (\mathbb{C} \cup \{\infty\}, (0, z \mapsto \lambda z), (\infty, z \mapsto \frac{1}{z}))$. Then you can sit down and apply the definition to see that $A_\lambda \circ A_\mu \cong A_{\lambda\mu}$. I want $\lambda \in \mathbb{C}^\times$.

Now let's come to graded vector spaces. I want them to be graded by an arbitrary number of copies of \mathbb{R} . I want them vector spaces over \mathbb{C} . So the objects are

$$A = \bigoplus_{\vec{v} \in \mathbb{R}^{n_A}} A_{\vec{v}}$$

where the dimensions of these $A_{\vec{v}}$ is finite. The morphisms are $\text{Hom}(A, B) = \text{Hom}_{\mathbb{C}}(A, \bar{B})$ where this is the product of the $B_{\vec{v}}$ rather than the sum. The monoidal structure has $A \otimes B$ is $\mathbb{R}^{n_A+n_B}$ graded.

The composition is partially defined because you end up in \bar{B} , it's not clear you can compose again. It's also not necessarily associative. I want functors between these that obey the following:

$C : ST \rightarrow GV$ which obey the following, I have to tell you why the grading. So first of all $C(1) = F$ which I want to be $\mathbb{R} \times \mathbb{R}$ graded, and I want $F(u, v) = 0$ if u, v are sufficiently negative.

Then I demand that the annuli I just defined are taken to, well, $C(A_\lambda)|_{F_{u,v}} = \lambda^{-u}(\lambda^*)^{-v}id$, and for this we need $u - v \in \mathbb{Z}$. So that's the thing we want to look at instead of Segal's version of a quantum field theory. So we want to get from Segal to correlators.

Let us suppose that $F_{0,0}$ one dimensional. I'll call this "uniqueness of the vacuum." Then I can pick out a preferred element there by seeing what it has to say to a sphere Ω with one outgoing boundary in standard parameterization. The vacuum is both the lowest energy and the only $SL_2(\mathbb{R})$ invariant in some theories, but here that might not be the same, so the

vacuum might not be lowest energy.

If I glue this Ω to a puncture, that puncture disappears. I want to pick an Ω' in the dual $F_{0,0}^*$.

Now I want to define my correlators $\langle \phi_1(p_1) \cdots \phi_n(p_n) \rangle$ where $p_i \in \mathbb{C}$ and $\phi_i \in F$. So it will be C applied to the sphere where ∞ is the outgoing puncture and the p_i are the incoming ones, with the chart $z \mapsto z - p_i$, and the standard $z \mapsto 1/z$ for ∞ . Then I apply this result to $\phi_1 \otimes \cdots \otimes \phi_n$ and then move to the ground field with Ω' .

Now the question, now we have defined what we mean by these correlators. Now how do they compute them? Then what you would say from the physics point of view is that your model has local symmetries that give rise to conserved currents. It's $G \in F$ so that $\frac{\partial}{\partial \bar{z}} \langle G(z) \phi_1(p_1) \cdots \phi_n(p_n) \rangle = 0$ for all ϕ_i and p_i . The dependence on the insertion point of that field is holomorphic. This notation is because you are putting a field ϕ at point p . This is a symmetric functor so the order here doesn't matter. This is a property of G , that if you put in any ϕ s things will depend holomorphically on z .

Here is an example: Ω , which does not depend on the insertion point. A more complicated example is the stress tensor, also called the stress energy tensor.

So we define the following morphism in our category ST , it goes from $0 \rightarrow 1$, I call it L_ϵ . It is $\mathbb{C} \cup \infty$ where the map is $z \mapsto \frac{1}{z} + \frac{\epsilon}{z^3}$. I will differentiate with respect to ϵ , and say $T = \frac{d}{d\epsilon} C(L_\epsilon)|_{\epsilon=0}$ which will be in \bar{F} . As an exercise, it's actually in $F_{(2,0)}$. You check that if you glue A_λ to L_ϵ then that's like $L_{\lambda-2\epsilon}$. So this is a conserved current. If you want to know more, I refer you to Huang.

If you have a classical field theory, then you get can get a conserved charge out of a symmetry. If I do a complex contour integral, it doesn't matter where I take it. That is the same property I have for a conserved charge, that it doesn't matter where I integrate in time.

[Conserved current in physics means not that it doesn't change but that the divergence is zero. It's an analytic condition]

I have changed the way the surface looks locally in the stress energy tensor. I don't know why this funny way.

Maybe the last thing I will do before the break is write down the Virasoro algebra. Now I am getting more sketchy. I define yet another surface, $P(\zeta) \rightarrow 1$, which is the standard Riemann sphere with inputs at 0 and ζ with standard coordinates. If I apply C I get something $F \otimes F \rightarrow \bar{F}$. Now I want to insert $\bullet \otimes T$ and see what that gives. I can write down a power series expansion and get

$$C(P(\zeta))(\bullet \otimes T) = \sum_{m \in \mathbb{Z}} \zeta^{-m-2} L_m$$

where you can check $L_m : F(u, v) \rightarrow F_{u-m, v}$. One verifies with a little star which I will explain (extra condition) that $[L_m, L_n] = L_{m+n} + \frac{c}{12} \delta_{m+n, 0} (m^3 - m)$. The central charge comes from annuli with funny local coordinates. The projectiveness kicks in and you get an anomaly. Then you want to extend your category so that the functor is no longer projective,

replacing ST with the determinant line bundle over the sphere with tubes raised to the complex power $c/2$. All of these moduli spaces, the determinant bundles are trivial. If c is an integer, an even one, it agrees with a power of the determinant bundle. Never mind. Then \hat{C} is no longer projective. I don't want to get into this in so much detail, but that describes the anomaly. If you're interested I refer you again to Huang. Maybe that's what I want to say now and I'll continue after break.

2 Ingo Runkel III

So far I have managed three pages of my notes per hour, now I will average five. We have done all of the things in the last lecture, we have seen how to get modes which satisfy the relations of the Virasoro algebra. I want to work out an example, the Ising model. I want to talk about the toroidal model. I'm not going to talk about how, but you can get the $c = 1/2$. Remember that at the critical limit you might get a CFT. I want to fix some parameters that you know for sure from the lattice model and see if it's determined. So what one can prove is that it produces the correlators of the Ising model at large distances. From the lattice, the central charge had to be $1/2$. It has this charge. The CFTs should be unitary because [unintelligible] is positive. We know we have an action of the Virasoro algebra on F . Then we have another copy with shifted grading. I'm asking how does F decompose. So first we study the irreducible highest weight representations with positive definite invariant inner product. There are three, R_0 , $R_{1/16}$, and $R_{1/2}$. You start with highest weight vector v in a Verma module, where $L_0 v = \frac{1}{16} v$, or whatever, and $L_m v = 0$ for $m > 0$. The other Virasoro algebra comes from taking ϵ 's conjugate in the stress energy tensor. This is a Verma module and you take a quotient. You can study what kind of functors one can write down. For the [unintelligible] the space of states is $R_0 \otimes R_0 \oplus R_{1/16} \otimes R_{1/16} \oplus R_{1/2} \otimes R_{1/2}$. graded by L_0 , \bar{L}_0 . Ω lives in the R_0 piece, since it's the only integer piece.

In the $R_{1/16}$ piece, we can take the product of the highest weight vectors and call it σ . So what is $\langle \sigma(z) \sigma(w) \rangle$. I could write this down easily with a scaling argument but I want to use a different approach. If I do $\langle \sigma(z) (L_0 \sigma)(0) \rangle$, that will be $\frac{1}{16} \langle \sigma(z) \sigma(0) \rangle$. I want to write this as a contour integral,

$$\oint_0 \frac{d\zeta}{2\pi i} \zeta \langle \sigma(z) T(\zeta) \sigma(0) \rangle$$

This will potentially have poles at 0 , z , and ∞ . So I deform a contour around 0 to one around z and one around ∞ . So that's

$$\left(\oint_z \frac{d\zeta}{2\pi i} - \oint_{0,z} \frac{d\zeta}{2\pi i} \right) \zeta \langle \sigma(z) T(\zeta) \sigma(0) \rangle$$

Through some difficulty, the contour around ∞ vanishes. Then I can, for z and ζ close, evaluate this to

$$\sum_{m \in \mathbb{Z}} (\zeta - z)^{-m-2} \langle (L_m \sigma)(z) \sigma(0) \rangle = \frac{1}{16} (\zeta - z)^{-2} \langle \sigma(z) \sigma(0) \rangle + \dots + (\zeta - z)^{-1} \langle (L_{-1} \sigma)(z) \sigma(0) \rangle$$

The other terms vanish, and so I get that this is

$$-\frac{1}{16}\langle\sigma(z)\sigma(0)\rangle - \frac{\partial}{\partial z}\langle\sigma(z)\sigma(0)\rangle$$

so we have differential equations (also doing the conjugate

$$-\frac{1}{8}\langle\sigma(z)\sigma(0)\rangle = \frac{\partial}{\partial z}\langle\sigma(z)\sigma(0)\rangle$$

$$-\frac{1}{8}\langle\sigma(z)\sigma(0)\rangle = \frac{\partial}{\partial \bar{z}}\langle\sigma(z)\sigma(0)\rangle$$

We already know we want an answer with $(\bullet)^{-1/4}$ but this illustrates the place of differential equations. This is a first order PDE whose solution is unique up to a constant, $\langle\sigma(z)\sigma(w)\rangle = \Lambda z^{-\frac{1}{8}}\bar{z}^{-\frac{1}{8}}$.

Let me make a comment about the paper from 84 that made this big. Similar arguments will let us figure out the three point functions up to a constant, but the four point functions only up to a function of one variable. But the true magic comes from that this is a Verma module quotient, not a free module, and in $R_{1/16}$, we know that $L_{-2} - \frac{4}{3}L_{-1}L_{-1}v = 0$. I can get a second order differential equation using this, and then the solutions will be exactly the [unintelligible] which are allowed. We have additional equations from the relations on the module, that's what makes this solvable.

Let me give a brief overview, I showed how to relate to physics, let me now continue the mathematics. The math from here, you take a subspace $V \subset F$ such that it makes sense to restrict C to V and such that dependence of the moduli is holomorphic. A result by Huang is that this can be described by a vertex operator algebra on V . Let me describe the data, not the axioms, of a VOA.

This is $(V, Y, 1, T)$ where V is a vector space graded by nonnegative integers, well, bounded below, with each $V_{(n)}$ finite dimensional, an element $1 \in V_{(0)}$, another $T \in V_{(2)}$, and $Y(\bullet, z) : V \rightarrow \text{End}V[[z^{\pm 1}]]$.

Then there are a bunch of axioms that I won't write down. Then Ω is like 1 and T is like T and Y is like the three point function and the category of these is the same as the category of such $V \subset F$. Then there's a theorem of Huang that tells us about the representation theory. So suppose V is a VOA that satisfies certain conditions that I'll omit, then the representations of V are a modular tensor category.

Then there's a theorem by Huang and Kang that says that if I have two vertex operator algebras satisfying the omitted conditions i through iii, then there is a functor $C : ST \rightarrow GV$ so that $V_L \otimes \bar{V}_R \subset F$, and this is the same thing as looking at commutative unital associative algebras F in the twisted tensor product of the representation categories of V_L and V_R with $\theta_F = id_F$.

There is a challenge, now that we've split the problem in two, there's a challenge by Kang and Ingo Runkel to say that these things with even more properties (genus one condition,

$V_L = V_R$, et cetera) are the same as simple special symmetric Frobenius algebras in a single one of these categories, $V_L = V_R$ so in $Rep V$.

The important thing is the Morita classes of such algebras. An open problem, to classify WZW models, take g , look at the affine \hat{g}_k , look at the representations, and what are the Morita classes of such algebras. The answer is only known for $sl(2)_k$.

After having said that, I have come to discussing whether morphisms have anything to do with defects.

[In the toy model, V_L is the complex numbers and you get the standard picture of Frobenius algebras, with algebras over the complex numbers.]

So what might one want to write down to give a morphism between CFTs? The first idea you might have would be, if you had a Hilbert space of states, suppose you had two functors, which give F and F' . One relation would be a linear map U that preserved the grading from F to F' . There should be more compatibilities. One way to think about this U , let me draw pictures in the Segal way. Then the linear map you might think of as $C(1)$ to $C'(1)$, but at the middle you do one step with U , but then there should be a compatibility condition, where if you put U on both legs that should have something to do with putting U at the waist instead.

You might think, that's like drawing surfaces. On one side I have one theory and on the other I have the other theory. In quantum field theory you might say this is not local because the circle is determined by any point on it. So say I want to cross in one place with U and another with V , is there a way to fill in between these?

So one wants a model. I go to my favorite helping model, the lattice model. What would that mean in the lattice? In the Ising map, I can go from one row to the other by inverting the spins. Remember the Ising model lives on a lattice. We have a function $\sigma : \Lambda_L \rightarrow \pm 1$, and an energy $E = -\sum \sigma_i \sigma_j$. I will now draw an line on the lattice that intersects lines but not vertices. So here I put on the edges cut by this a $-J_{ij}$ instead of J_{ij} , where J_{ij} is usually 1. Can I make sense of doing this in one area and not in another? In the lattice model I can just stop flipping the spins. Across one line I reverse the interactions. Then I can apply that symmetry locally. Here I am computing $\frac{1}{z} \sum \sigma e^{-\beta \sum J_{ij} \sigma_i \sigma_j}$ Another thing, this is equal to changing the path of my dotted line, it's the same, it only depends on the endpoints, not where the line runs. The z is a scaling factor where all J_{ij} are 1. So this is the same as $\langle \sigma_i \sigma_j \rangle_{\tilde{\beta}}$, where $\tilde{\beta}$ is some observed function given by $-\frac{1}{2} \log \tanh \beta$. So a spin correlator at low temperature is the same as the disorder correlator at high temperature, where the i and j on the right hand side are the insertion points.

This gives you the following idea. Look at the category, a different one, to start the functor. Look at surfaces with homotopy classes of lines. In the Segal way you would look at Riemann surfaces with defect lines, like this. The endpoints as they are in the pictures I've just been talking about come from shrinking the boundaries to points. If you do this topologically, with no conformal structure, it looks like a planar algebra to me.

So then there is a remark. The spheres with tubes sit in ST_{def} , which has the lines. Then

this will be more difficult. They're not numbers, they're lists of plus and minus signs around each point. Now you have to give an infinite $C(\cdot)$, then $C(+)$, $C(-)$, and so on. You have to give an infinite amount of different vector spaces. Fortunately, one can get examples of such things. We think we can prove, if $C : ST \rightarrow GV$, and this functor comes from a special symmetric Frobenius algebra in the representations of a VOA, then a choice of an $A - A$ bimodule gives an example of $C : ST_{def} \rightarrow GV$, which is joint with Fröhlich, Fuchs, Schweigert and myself, and in progress with Kang.

What would you do with two theories? You'd two color your operad. You'd choose an $A - B$ bimodule, and the CFTs with the property that $V \otimes V \hookrightarrow F$ give a two-categorical structure, where the objects are special symmetric Frobenius algebras, 1-morphisms are $A - B$ bimodules, and 2-morphisms are morphisms of bimodules.

I have only talked about the Euclidean approach. You can do this in Minkowski space, and that is different, with nets of operator algebras, and you're led eventually to the same objects. So you need the representation theory of certain conformal nets on the circle, and then you eventually need similar things as this. I think I said enough, no? Thank you very much.