Operads

Gabriel C. Drummond-Cole

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1 Bruno Vallette

[Good morning, we didn't do a round of introductions because we had too many. I have made a list of participants. I'll put it here. Let's go on now.]

So during this last talk, let me recall what we were doing yesterday. You had a category of algebra modelled with a properad. Now to model it up to homotopy, I replace it with a cofibrant resolution, which I choose to be a quasifree resolution. I define a P_∞ algebra, or homotopy P-algebra to be an algebra over P_∞ (the cofibrant replacement). We know that if we take two of these, the homotopy category of algebras is the same. If you want to do homotopy, you look for a cofibrant replacement; the categories of algebras are not the same but they are homotopic. If you ask for minimal replacements, then they are unique, the d, which is characterized by explosions d_n into n vertices, you ask that d_1 is zero, so then the underlying category will be the same.

The goal today is to come up with quasifree replacements. Then I will give four equivalent definitions, and for each we will see what we can do. I want to be lazy and use a definition that will make things easy for each thing I want to do.

So now I need homological algebra for operads. Let me recall what we know on the level of algebras. If we start with C, A, a dg coalgebra and algebra, we can consider $Hom_k(C, A)$, where we find the convolution product (which is associative) $f \star g = \mu \circ (f \otimes g) \circ \Delta$, which is a binary product. Now you get that this forms a dg associative algebra, where you can consider the Maurer-Cartan equation $\delta\alpha + \alpha \star \alpha = 0$. We'll be studying the set of solutions to this equation. The set of solutions is called the twisting morphisms. Now we can ask the question, what is TW? It is a bifunctor to Set. Can we represent this? We're looking at $Hom^{-1}(C, A)$, in which we want to look at $Tw(C, A)$, which we want to represent on the left and on the right. So we want a construction on C to give an algebra, so that $Hom_{alg}(?C, A) \cong Tw(C, A) \cong_{coalg} (C, ?A)$, and a construction on A to give a coalgebra. So I need to take the free algebra on C and because of the degree -1, I need a desuspension, so $T(s^{-1} \otimes C)$. I want to deal with nonunital algebras and coalgebras so I don't have to deal with the details of killing the unit. On the coalgebra side we do the dual thing, so we take the cofree coalgebra on the suspension of A, $T^c(s \otimes A)$, so so far we are close to the solution.

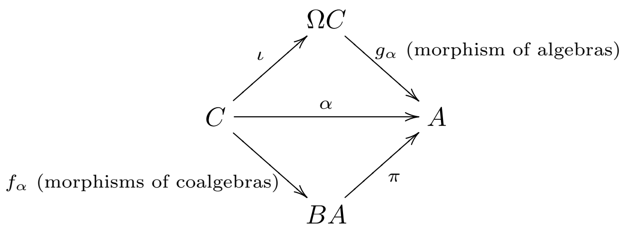
The difference is that there is no dg. I need to define the differential, and the way I did it yesterday, it is the exact same as the bar construction on the right and on the left the unique derivation extending the coproduct, and that is the cobar construction. So here we have bar and cobar. If you believe, I will not do the proof, the morphisms here, they commute with the differential. There is nothing to do here.

$$Hom(\Omega C, A) \cong Tw(C, A) \cong Hom(C, BA)$$

[Are they the same relation, the homotopy relations here on the three lines? What is it in the middle?]

Yes, and the gauge action.

Okay, so now we have an adjunction, and so if I take C to be BA, let me take that to this case. So here on the right I have id_{BA} which corresponds to a twisting morphism $BA \to A$, and then a map $\Omega BA \to A$, which is the counit of adjunction. If i do the same thing with $A = \Omega C$, then $id_{\Omega C} \leftrightarrow C \to \Omega C$ a twisting morphism, to $C \to B\Omega C$, the unit of adjunction. So these twisting morphisms are unique in which sense? They factor through morphisms of algebras and coalgebras:



So I have quasiisomorphisms on the left and right, how do they compare to the things in the middle? Well, let me look at $C \otimes A$. I have $d_{C\otimes A}$ but I want to consider a new differential. If you have a twisting cochain α, and

$$\bar{d}_\alpha = (id \otimes \mu_A) \circ (id \otimes \alpha \otimes id) \circ (\Delta \otimes id)$$

[Take chains on a space, and that's a coalgebra. You can think that A is a topological monoid, and then you can construct B of that monoid. If you look at that monoid. Apply B and you get a classifying space. Take a space, apply Ω and you get a monoid (with inverses)]

You get this twisted differential, with $d_\alpha = d_{C\otimes A} + \bar{d}_\alpha$, which is called the twisted tensor product. When α is a twisting morphism, this d_α squares to zero. So one question, when is this acyclic? this is when α is called a Koszul morphism. It has something to do with Koszul duality theory. Now the theorem is that the quasiisomorphisms of algebras or coalgebras are equivalent along our adjunction to the Koszul morphisms.

[This is a formula for the principle bundle, and when the bundle is contractible, the base space is the universal space]

The lemma first is that π and ι are Koszul morphisms, so that $BA \otimes_\pi A$ and $C \otimes_\iota \Omega C$ are acyclic.

Theorem 1 *(Brown)*
The following assertions are equivalent (in the presence of a weight grading):

1. $C \otimes_\alpha A$ *is acyclic*
2. $f_\alpha : C \to BA$ *is a quasiisomorphism*
3. $g_\alpha : \Omega C \to A$ *is a quasiisomorphism*

We have now solved a syzygy problem. You kill the homotopy step by step, you do everything at the same time. It involves finding a projective resolution in the category of A-modules, and calculate Tor. On the bottom we want to find a cofibrant resolution on A. This is a second syzygy problem. A resolution in the category of dg associative algebras. So the corollary is that $\Omega BA \to A$ and $C \to B\Omega C$ are quasiisomorphisms. These resolutions always work. This corresponds to the Bordmann-Vogt resolution. This gives a cofibrant replacement, which is huge. The opposite, $\Omega C \to A$ is smaller but what is C and what is α?

Let me extend this to operads, properads. You've seen yesterday what a properad is, it's a monoid in a space. A coproperad is a comonoid, you have maps that decompose. You consider maps $Hom(C, P)$, which forms a convolution properad. This induces a product which induces a Lie algebra. You actually want $\mathbb{S}$-invariant maps, and the convolution properad structure does not survive, but still you have the differential graded Lie algebra structure surviving. So now you can copy-paste the blackboard, and get $\partial\alpha + \frac{1}{2}[\alpha, \alpha] = [$, and these of degree -1 are called twisting morphisms $Tw(C, P)$.

So now for the adjunction you need the free properad $F(s^{-1} \otimes C), P) \cong Hom_{\mathbb{S}}(C, P) \cong Hom(C, F^c(s \otimes P))$. What was the differential on the bar construction of A? Here there is a point. The bar construction of A had $BA = T^c(s \otimes A)$ and d was the unique coderivation extending the product of A. In the operad, we take the cofree cooperad, so these are trees with internal vertices labelled by $s \otimes P$, and they defined the differential using the graph homology of Kontsevich, and when you contract, you form the composite. For properads, this doesn't work. Why? You cannot use edge contraction because of [picture]. Go back to the universal property. It's the universal coderivation extending the product. You do the product anywhere you can. You can compose these two [picture] or these two, but not these because they produce cycles. Just to tell you, now you can extend that, this is the only subtle thing, that now you have

$$Hom(\Omega C, P) \cong Tw(C, P) \cong Hom(C, BP)$$

and you can copy everything, it all remains true. But what is the twisted tensor product? You just need $\boxtimes$ instead of $\otimes$. So the Brown theorem, everything is still equivalent, that $C \boxtimes_\alpha P \xrightarrow{\cong} I$, $C \xrightarrow{\cong} BP$, and $\Omega C \xrightarrow{\cong} P$

Now we get a cofibrant replacement functor for P-algebras, and what we were looking for is the third condition, to see quasifree resolutions of the properad P along g_α, the map from the twisting morphism α.

Once again, we want to resolve, we take the generators, homotopies for the relations, and this will solve the syzygies once and for all. By abstract properties, you will get all the syzygies. So to get a smaller resolution you look for an acyclic twisting cochain. Bar cobar, you get a very big resolution. For Frobenius bialgebras, C is really a mess, so maybe use this big one. This is a nice theory but it is not so far what we are looking for. This gives a nice method to find these resolutions. It's exactly where Koszul dualities comes in.

Let me talk about homogeneous quadratic. A properad is homogeneous quadratic if $P = F(V)/(R)$ where $R \subset F(V)^{(2)}$. Let me give you an example. Ass is generated on one generator with a free S_2 action modulo associativity, which is quadratic. Commutative algebras is similar, so is Lie, where Jacobi is quadratic, Gerstenhaber, biLie, $k[\Delta]/\Delta^2$, and then in parentheses (Frobenius and involutive Lie bialgebras). Koszul duality says that for C take a construction $P^{\text{¡}}$ the Koszul dual coproperad, and then you have a nice Koszul morphism $P^{\text{¡}} \to P$. It's easier to prove that something is acyclic than to do something is a quasiisomorphism. To define $P^{\text{¡}}$ is by universal properties but I will tell you how to cheat.

If the space of generators and relations is finite dimensional, I can dualize always a coproperad to a properad $(P^{\text{¡}})^* = P^!$, and then the dual is also quadratic. The generators are dualized. There are signs but for the specialists please say nothing. Take the free algebra on the dual space of generators $F(V^*)$. Then $Ass^! = Ass$, $Com^! = Lie$ and $Lie^! = Com$. For Frobenius and involutive biLie, I don't know how to show that they are Koszul.

Just a few words, how to go beyond. There are two ways, if the relations are not homogeneous, it means that the properad is $F(V)/R$ where the relations can have linear and quadratic terms, the basic examples are the Steenrod algebra, the universal enveloping algebra of a Lie algebra, and a BV algebra. Linear compositions are related to composites. I should mention names. Priddy, Ginsburg Kapranov Cohen Jones, [unintelligible].

Consider a quadratic properad out of that by killing the linear part. Then add an internal differential which is defined by an algebraic property. The Koszul dual is now a dg coproperad. Since I'm short on time, we developed with Merkulov homotopy Koszul, then if you have higher, then it is a homotopy coproperad. Then you know exactly the space of generators and how to get the structure on the dual. Here you have the construction of bialgebras (Hopf algebras). We have concrete methods to make these explicit. How do we do that?

Now we have explicit resolutions, and we can study P_∞ algebras. Let me apply the adjunction you saw before. What is a P_∞ algebra (in the Koszul case where we have $\Omega P^{\text{¡}} \to P$)? Well $Hom(\Omega P^{\text{¡}}, End_A) \cong Tw(P^{\text{¡}}, End_A) \cong Hom(P^{\text{¡}}, BEnd_A)$. So $Hom(P^{\text{¡}}, Hom(A^{\otimes n}, A)) \cong Hom(P^{\text{¡}}(A), A) \cong Coder(P^{\text{¡}}(A))$ and I can add to the brevious list square zero coderivations on $P^{\text{¡}}(A)$. One comment. This works for coproperads. It's not an algebra but a comodule in the dual case. What does not work if it is not Koszul? There is no strict adjunction, so I can't get $Hom(P^{\text{¡}}, BEnd_A)$ or square zero coderivations. It's also a homotopy coproperad so the twisting morphisms are in an L_∞ setting, not a Lie one.

So okay, what is each definition for? The square zero coderivations are for homotopy theory, ∞ morphisms. The first definition allows us to do something over a cofibrant operad. You take a resolution of BV. Then we have the explicit definition. Since it's cofibrant, well, we know that this is framed little disks, and this is formal, so we can lift homotopy BV to framed little disks. You know that this is homotopically equivalent to Riemann spheres, and so this is a particular case of the properad of Riemann surfaces, which are TCFTs. So a TCFT has the structure of a homotopy BV algebra. Then in that you have Riemann spheres. So homotopy BV captures a small part of a TCFT. There is no topology here at all. You have explicitly the relations. The third definiton, that gives easy transfer of structure. Each time you have S a strong deformation retract of A, and a P_∞ algebra structure on A, how do we transfer to S? We want to take $Hom(P^{¡}, BEnd_A)$ to $Hom(P^{¡}, BEnd_S)$, and you have a map $BEnd_A \to BEnd_S$ from the strong retract, so you push under this map to transfer structure. With this definition it's easy. Then you understand the Kontsevich Soibelmann formula conceptually. You also have explicit formulae for anything.

Then two things, the twisting morphisms, if you want to do transfer beyond the Koszul case you can do the twisting morphisms for transfer of structure. You start with a P_∞ structure on A, $\alpha \in Tw(P^{¡}, End_A)$ and you want something in $Tw(P^{¡}, End_S)$ in $Hom_{\mathbb{S}}(P^{¡}, End_A)$ and $Hom_{\mathbb{S}}(P^{¡}, End_S)$. You have no strict maps of Lie algebras but you have a weak map. You can get an L_∞ morphism from the map of $BEnd_A$ to $BEnd_S$. So this morphism preserves Maurer Cartan elements. Now we can go beyond the Koszul case, this is a homotopy coproperad. So now these are L_∞ algebras instead of Lie algebras and the proofs are the same. This gives the explicit formulas for transfer of structure of anything up to homotopy. Then this is the good Lie algebra to do deformation theory. The deformation theory sits here because the Lie algebra is very particular. The dual is weight graded. The ideal was homogeneous, so we can count a weight on the free construction, and we get a weight graded Lie algebra. The Maurer Cartan equation splits, and we get on vertex operator algebras a homotopy BV algebra. This is an example for obstruction theory. Twist the Lie algebra and you get a deformation. The chain complex is the [unintelligible]complex, so there is a nontrivial L_∞ structure there. This cohomology theory can be understood as being represented by a cotangent complex that I think will have very nice properties.

Thank you very much.

[In your first lecture QFT was in one corner of the triangle. Did you mean?]

It's this TCFT connection, then because this is a monoid, Riemann surfaces, you can relax up to homotopy, and get a homotopy topological conformal field theory, and maybe the link to vertex operator algebras.

[I've been trying to understand quantum field theory so I say "what is the algebraic structure of correlators?" Mathematicians should have something there. What is the structure of these? Someone says that it's an ∞ structure spread out over spacetime. He's a real physicist and feels that there's an algebraic structure, an algebraic and analytic aspect. These ideas are germane. It's not there yet. You haven't got all the traces, degree zero terms]

Just to mention the traces, you can add wheels to this graph. If you add wheels you get a BV algebra that models this structure of deformations.

[Kevin was on a Riemannian manifold, where he had to renormalize infinities, which was close to this and also close to physics. More and more there's a feeling that an effective theory is a transfer of structure. Those ideas are to be fit together.]

[This needs the higher category part too.]

[You have semisimple reductive parts and then the nilpotent part. You develop the homotopy theory, which has a semisimple part for the fundamental group. These are preliminary to discussing the nilpotent part which has to be fused together.]

[I read your '78 paper so I know what you're talking about]

[First schemes were sets and then they had nilpotent elements, and I'm sure Jacob can give meat to that idea.]

[Let's thank Bruno again.]

[For those of you interested to this, we are organizing a school in Luminy, the third week of April, and then we invited several researchers. So you have people using these in many fields. They will explain what they are doing. Then the next week a conference.]

[Let's start again at 11:15.]

2 Michael Sullivan

[There's going to be a thing on TFTs in Northwestern, May 25-29 is a workshop with Ben-Zvi, Lurie, Toen, and that's the wrong week, the workshop is the previous week, that's the week of the conference. We have funding so people should email us and apply. The workshop means that there ar minicourses.]

I want to thank Peter for organizing. I'm definitely learning a lot. I wish I weren't talking after Bruno's third talk. I couldn't listen because I was wondering whether what I was going to say was correct. The title is Open String Topology and knots. It's joint, in progress, with Dennis Sullivan. Basically a quick outline. First I'll give a tiny little overview for what one would hope for. Then I'll do some examples in the case of knots. Then I'll give a theorem. I'd like to discuss the proof, and then hopefully Legendrian symplectic field theory. That means a couple of definitions and a couple of drawings. Okay, so all right. We're going to start with a smooth knot K in $\mathbb{R}^3$ which comes equipped with a framing, and we're going to, open string topology, what we're looking for here is the space of paths $\{f : [0,1] \to \mathbb{R}^3 | f(0), f(1) \in K\}$. So it begins and ends in the knot and can pass through the knot. This space is homotopy equivalent to $S^1 \times S^1$, where these are just the start and end points. You can just homotope to the space of straight paths. The goal is to define some string topology operations on this space, interpret them as an algebraic structure, and then extract nontrivial knot invariants from that. That would be the goal.

There are a lot of people here familiar with string topology and the problems that arise in different cases. The first thing you might try would be to do the open version of the Chas-D.

Sullivan closed string product ('99). In the case like this, if you have strings starting and ending on a knot where the start point of one coincides with the end point of the other, you product them to get a single string, but it's not too hard to see that this has nothing to do with how K is embedded in $\mathbb{R}^3$. The product doesn't work, it's a trivial knot invariant. You might think, okay, there's no knot theory, because when you try going to the coproduct, that has, as Dennis likes to say, an anomoly in it that doesn't commute with the topological boundary, so let me move to examples. What we're going to do is take the coproduct and adapt it so that it passes to homology. So now I have to give a disclaimer. This could be a different well-known knot invariant. So I'm not claiming this is new. We start with, K is a framed knot, and choose K a knot projection such that the framing is the blackboard framing. You can do this with Reidemeister I moves. Then orient K and pick some point on it and consider the S^1 family of straight lines starting at x_0 and ending at x.

Then calculate the following contributions: If I have a picture like this: I can think of the framing as a section in the normal bundle, imagine I have a family of strings running like this, and then you assign a $+1$ to running into the knot through the section. On the other hand, you might have a string with an internal intersection where the string hits the knot, which is also $+1$. I think the last one is -1, where the start of the string passes through the framing going outward. Call this type C. These are switched if you are in the opposite framing.

This is not a full projection, it's flattened. I did it this way for the blackboard framing so that we could look at this picture.

You can perform this computation, but essentially you get no contributions af type C because the framing is on the other side of the knot, so to speak. In the unknot you get 4 type A contributions. The invariant should be $A + C - B$ So we have three different numbers for three framed knots.

For any knot, the first homology is one generator, and the first homology has two generators, b_1 and b_2. In the second step, I was picking out b_1, a representative for this homology class. b_2 would be fixing the end point and varying the start point. Type B move should be a coproduct (an open one) so morally to get this, if it worked, we'd have a map from homology of strings, if the input is k dimensional and the output is $i-k$ dimensional, you're intersecting a little 1 manifold with a k manifold and getting a $k-1$ dimensional manifold [unintelligible].

Now let me get to, well, $[\Delta, \delta] \neq 0$ implies that Δ is not well defined on the homology. The solution is to introduce another operation $\tilde{\Delta}$ so that the sum of these commutes with ∂ and is still coassociative. Every time we see the string passing through the knot we cut and consider as a pair of strings. Cut instead at type A to the constant string. So $\tilde{\Delta}$ will be intersection of tangent vectors with a 2 dimensional half plane distribution spanned by the framing vector and knot tangent vector. You might start with a closed family of strings, and ask, where does this thing intersect the half-plane. This has some boundary, which is where that was tangent to the knot. Okay, maybe I'll skip the operad story.

Let me state a simpler version of the main theorem. Let T be the tensor algebra generated by the homology of the knot and K a framed knot. Then there exists some D depending on the knot, which decomposes on the tensor algebra and is square zero. If you have an isotopy, then these are isomorphic structures, with the same D_2, so they are isomorphic A_∞ algebra invariants.

Let me skip to the sketch of the proof or ideas of the proof. The idea comes from Fukaya-Ono-Joyce and Fukaya-Ono-Odo-Oh. Suppose I have two submanifolds and I want to intersect them but I don't know that they're transverse. What does it mean to take this intersection?

[Oops, I started calculating invariants and stopped paying attention]

Thicken A and B and intersect, but retain thee original information with a section in the obstruction or normal bundle. Then take the Whitney sum. [unintelligible]

The dimension of the thickening minus the dimension of the fiber is the formal dimension of the Euler object.

My interest in this comes from rational Legendrian symplectic field theory. This counts J-holomorphic curves with Legendrian boundary in a symplectization or contact manifold. The picture is the following. You have a disk here, and then by plusses and minuses you want to count maps into $M \times \mathbb{R}$. I'm not going to say Legendrian versus Lagrangian. The master equation says that if you have $\delta\mathscr{M}$, this is by Gromov compactness contained in $M'\#M''$, and on the other hand it contains it by the gluing theorem. So the question is what is the gluing. Once you answer that question you can impose structure like an algebra over an operad. If there's no boundary it's called symplectic field theory. At the boundary you have the problem of bubbling. This one causes a problem from composition's point of view. Here you don't have a sign. That's one problem. What's the algebra. Another problem is string topology, because you are intersecting chains of strings. The final question is that there's a natural way of embedding [unintelligible]in [unintelligible]. So all questions in differential topology can be restated in symplectic Lagrangian homology. So what does $ST(K)$ equal in the SFT of its lift? That was my original interest.

3 Thomas Tradler

I'm going to talk about higher Hochschild complexes. I hope they fit into the theme of these conferences. The reason I say that I hope it fits is that I haven't actually shown it. For motivation, a goal, something I would like to do that I can't quite do yet, I would like some higher dimensional TQFT. Let me be very vague. Nothing here is set in stone. What is the idea here? If you look at Jacob and Mike's definition, they look at tensor functors from some category *nBord* into a target category $\mathscr{C}$. If this has objects algebras and 1-morphisms bimodules, then intuitively you associate an algebra to a point and a morphism will be something like a bordism to two points, and by fairly general arguments, the circle has to be sent to the Hochschild complex. This is $CH_.(A)$, and you put A in a circle. There is a clear correspondence here to the circle, it's not very precise. I want to argue that this interval should be sent to putting A on the edge, with two endpoints. This is like the two sided bar construction. Let's say you want higher things between these two things. For a surface, what do you put over here? That's exactly what the higher Hochschild complexes can provide. That's the idea.

So I want to be, the idea here is to use Pirashvili's higher Hochschild complexes. He phrased this. There's a paper by him in 2000 which is called "Hodge decomposition for higher Hochschild complexes." The basic idea is the following. First of all I'll fix a field $\daleth$ of characteristic zero. Various pieces generalize. Everything is going to be over this $\daleth$. Given A a commutative associative differential graded algebra with unit, I can do the following. I can get a functor from a category $Fin \to dgVect_{\daleth}$. The objects of Fin are finite sets and the morphisms are maps of finite sets. Over on the right the category is differential graded vector spaces over $\daleth$. Let's say I have a finite set, then I associate, map S to $CH^S_\cdot(A)$, which is $A^{\otimes S}$. What do you do with the morphisms? If I have a map $S \to S'$, then I want to get a map $f_* = CH^f_\cdot(A) : A^S \to A^{S'}$. If you have $a_1 \otimes \cdots \otimes a_n \mapsto b_1 \otimes \cdots b_{n'}$, where I look at the preimages of points, and say $b_i = \prod_{f(j)=i} a_j$. Of course nothing might land here, so if nothing lands here I just use the unit.

This is sort of everything that is going on here. So I want to point out here, you need A to be commutative, you don't know in which order to multiply, so any order has to work out. Okay, so, this defines the functor.

All right, so we want to get to Hochschild, I want to talk about pointed finite sets. If (M, d) is a commutative module, right and left module over A, then we get an induced functor from pointed finite sets to dg vector spaces over $\daleth$. If you have a pointed set $\{0, \dots n\}$ with 0 the basepoint, then you map 0 to the module and everything else to A. Similarly you have induced maps. If you have a map of pointed sets, the fat point is the base point [picture], the basepoint goes to the basepoint and maybe other things, so this maps $m \otimes a_1 \otimes \cdots a_n \to m' \otimes b_1 \otimes \cdots b_{n'}$ where again I have preimages of points. Again you have this induced functor and everything works out.

How do we do something interesting? The next step is to look at simplicial pointed sets. Recall that a simplicial pointed set is given as a functor, a contravariant functor from a category Δ into finite pointed sets, where Δ is the category whose objects are sets of the form $\{0, \dots, n\}$ for $n \geq 0$ and morphisms nondecreasing maps $f(i) \geq f(j)$ for $i \geq j$. This is the short definition but then you have to work to find out what this means. Then we get a functor from Δ to dg vector spaces, and we get something which by definition is a differential graded vector space, and that's the higher Hochschild associated to that simplicial pointed set.

Explicitly, what does it mean to have a pointed finite set. You have to, for each $n \geq 0$ you get a finite pointed set. For the morphisms, if you analyze this, it's generated by maps called the degeneracy maps and then the boundary maps d_0 and so on, and also the degeneration maps s_0, s_1, and so on. These are tantamount to the maps in the normal presentation of simplicial things. The geometric realization will be a pointed space. So we get $CH^{S_j}_\cdot(A)$ maps to and from $CH^{S_{j+1}}_\cdot(A)$. When you have a differential graded simplicial vector space you can define $CH^{S_\cdot}_\cdot(A, M) := \bigoplus CH^{S_j}_\cdot(A, M)$ and then you can define $b = \sum (-1)^i (d_i)_*$ and then $b^2 = 0$ So then $HH^{S_\cdot}_\cdot(A, M) := H_\cdot(CH^{S_\cdot}_\cdot(A, M), b + d_{A,M})$.

This is the Hochschild complex of A with respect to $S_.$ with M. This is functorial in $S_.$, A, and M, and if $S_. \to S'_.$ is a quasiisomorphism, then the induced map on $HH^S_.(A, M) \to HH^{S'}_.(A, M)$ is a quasiisomorphism. Now you have to find these complexes for any set. So one thing to connect it to regular homology is to do Chen's iterated integrals. Let A and M be the de Rham forms on a compact oriented manifold. What do I get? A Chen iterated integral map that goes from $CH^{S_.}_.(\Omega X, \Omega X) \to \Omega(X^{|S_.|}$. Then as usual, you have an evaluation map $\Delta^j \times M^{|S_.|} \to M^{S_j}$. You integrate along the form to get $M^{|S_.|}$. In some cases it's a quasiisomorphism, if we assume that X is at least connection up to the dimension of $S_.$, then this map is a quasiisomorphism.

If you put in the circle, you get the old result, the usual Hochschild homology is isomorphic to the homology of the loop space. One last thing before these examples. These were Hochschild homologies. Now we can define the Hochschild cohomology with the same data to be the dual space $HH^{\cdot}_{S_.}(A, M)$, well, cochains, to be $Hom_A(CH^{S_.}_.(A, A), M)$.

Okay. So, um, I want to do a bunch of examples. So far it seems like general nonsense. Let me start with an interval. The simplicial set, you take I_j to be the set $\{0, \ldots, j+1\}$. The boundary map has fewer points. It collapses one of these at a time. The d_j collapses between d_j and d_{j+2}. So the Hochschild chains are $\bigoplus_{j\geq 0} M \otimes \underbrace{A \otimes \cdots A}_{j} \otimes A$

The induced map multiplies them. In particular, this was the two-sided bar construction.

If you do the circle, it's similar, except you collapse the endpoints, and then you have M with A sitting around it, that's, well, s^n_j just maps the points to each other and it's exactly the Hochschild complex with the usual Hochschild differential.

Maybe one more example before I go to higher dimensions.

Even in one dimension, if I take a directed graph, I need a commutative algebra.

[Picture example of pointed graph with genus]

It's really interesting to go to higher dimensions. Let's look at the square. This can be the product of $I_.$ and $I_.$. The j simplices are just $\{(k, \ell) | 0 \leq k, \ell \leq j+1\}$. The Hochschild complex is $CH^{I^2_.}_.(A, M) = \bigoplus M \otimes A^{\otimes (j+2)^2 - 1}$. You now need to move the differential. The first differential d_1 would multiply together any pairs next to one another, in either factor.

I really think of this as a grid, of course, in some manifold where forms are sitting. The boundary has to make sure that each one knows about the next one. I can next go to the torus, the sphere, higher cubes, any manifold. The next example would be the torus. All are these are examples of just the part going to pointed finite sets, and the rest follows from the functor to dg Vector spaces. So you identify the last edges with the first edges, so you have $(j+1)$ rows and columns. The last piece of the differential maps the last and first columns and rows together. This gives a Hochschild complex with the mapping space of the torus into X. This is a nice finite model with finitely many ponits. You could look at what operators one has. Here now on the torus you have other operations. Chris Douglas analyzes this. Then you can do the sphere, S^2, and my simplicial model. I can do this for A

the forms and M also the forms. One example I wanted to do where they're not the same. $CH_{S^2_\cdot}(\Omega_{dR}X, \urcorner)$, well, the module is the base point. If this is X, then this calculates, the homology of this is the homology of the based iterated sphere space $\Omega^2 X$. If I allow $\urcorner$ to be any de Rham form, it would be the free space. That's why I wanted to distinguish A and M. I wanted to mention that Ginot showed how to get, in this case, what the Gerstenhaber algebra structure is on this space, you can, whatever structure you know exists, you can try to realize this in a nice simplicial setting, and you get [unintelligible]. It's interesting that you can make the BV Δ and there's no finite map between these two.

The last example, for any topological space, and I'm not sure what I should say, maybe call it Z, we have the canonical simplicial set $\Delta_\cdot(Z)$ where $\Delta_j(Z)$ are continuous maps $Maps(\Delta^j, Z)$, which is not a finite set. You can still define the Hochschild for that. For an infinite simplicial set, take $CH^{S_\cdot}_\cdot(A, M)$ by a direct limit construction, where you take the limit over finite simplicial subsets, and you take the direct limit of the objects $CH^{S'_\cdot}_\cdot(A, M)$, so in general you can always apply this and get its Hochschild. Z should have been a pointed topological space. The constant map to the point is the basepoint in $Maps(\Delta^j, Z)$. As Mike said, this shoud really be blob homology. You have all these relations, like, one can define a shuffle product on $CH^{S_\cdot}_\cdot(A, A)$ which is a standard construction on differential graded simplicial spaces. This corresponds to the wedge product or cup product, corresponding to the wedge product under the Chen map, and we have a pushout property, where if $W_\cdot = S_\cdot \cup_{U_\cdot} T_\cdot$, and I have $U_\cdot$ inside $S_\cdot$ and $T_\cdot$, and then the induced map of Hochschild complexes

$$\begin{array}{ccc} CH^{U_\cdot}(A,M) & \xrightarrow{f} & CH^{T_\cdot}(A,M) \\ \downarrow g & & \downarrow \\ CH^{S_\cdot}(A,M) & \longrightarrow & CH^{W_\cdot}(A,M) \end{array}$$

So if f or g is an injection, then yuo get that this last is $CH^{T_\cdot}(A, M) \otimes_{CH^{U_\cdot}(A,M)} CH^{S_\cdot}(A, M)$.

How would you get an evaluation map like in Kevin's talk? $eval : C_*(Diff(|S_\cdot|)) \otimes CH^{S_\cdot}(A, M)$ to $CH^{S_\cdot}(A, M)$. If we take $S_\cdot = \Delta_\cdot(Z)$, anything would act. I'm out of time, I'll stop. I should stop, this justifies a connection, although there are many things one would have to check.

4 Ingo Runkel

I called this aspects of conformal field theory because I couldn't think of a good title. I am Ingo Runkel. I started out as a physicist and now I am trying to become a mathematician. I was asked to say how conformal field theory relates to physics. How do I get a conformal field theory from a quantum field theory or from lattice models. In saying this I will say the word correlator. Then I will try to say how to get from Segal's CFT to correlators. Today we will probably get to 1.5 or 2.3 or somewhere. I want to talk about deriving differential equations for correlators. If there is time for a third bit, I want to say how if one wanted to talk about morphisms between CFTs, a good way to talk about it is with defect lines.

[Do you have an a priori motivation for why correlators satisfy differential equations?] It's like a connection on the moduli space, or the theories you are most interested in is where you can compute them, and there they satisfy differential equations.

Let me start then. What is the idea? Conformal field theories do not appear in nature; they are approximations. Unless string theory has to do with nature, they don't appear. It describes either a quantum field theory at short distances, which is synonymous with high energies, the ultraviolet behavior, or at long distonces, low energies, or the infrared region. So that's why there's no length scale.

They also want to describe lattice models at large scales. That can be in any dimension. Lattice models at large distances you can't tell a lattice from a continuuum. In a continuous thing you expect this to be true at both ends but close up in a lattice you can tell that the lattice is discrete.

This lattice model is only interesting at critical points. I will want to write down the Ising model in two dimensions later. The reason I'll only talk about two dimensions is because there you can say the most concrete things.

So the first example I want to look at is the free boson. This takes a field $\phi : \mathbb{R}^2 \to \mathbb{R}$, with the Euclidean metric on $\mathbb{R}^2$, and there's an action S which assigns to ϕ a number

$$\frac{1}{8\pi} \int d^2x((\partial_1\phi)^2 + (\partial_2\phi)^2 + m^2\phi^2)$$

Now we see our first correlators. You say $\langle\phi(x)\phi(y)\rangle = \int D\phi\phi(x)\phi(y)e^{-S[\phi]}$. This integral doesn't make much sense. Then we divide this by the partition function $\int D\phi e^{-S[\phi]}$. Both sides are infinite and the answer when regularized, is $2K_0(m|x-y|)$ where K_0 is a modified Bessel function.

There's an example of a two point function defined by a path integral. The correlator computes, if I were to do an experiment where I measure the value of the field at position x and position y, what is the average value that I will get. Now let's look at this at short and long distances. So let's rescale the coordinates by a factor λ.

I would like to do a baby version of the renormalization group. I want to look at $\langle\tilde{\phi}(x), \tilde{\phi}(y)\rangle^{new} = \langle\phi(\lambda x)\phi(\lambda y)\rangle_m = \langle\phi(x)\phi(y)\rangle_{\lambda m}$. So this is a one parameter family. What happens at the ends? When we take λ to 0 and to ∞, what do I get? These are the ultraviolet and the infrared conformal field theories. Now we just take λ to 0 and ∞ and look at the asymptotic behavior of the Bessel function.

This is not good to do with ϕ itself, but $J(x) = i\partial\phi(x)$ where ∂ is the holomorphic derivative $\frac{1}{2}(\partial_1 - i\partial_2)$. Then I can take $\langle J(x)J(y)\rangle$, and then I can take $\lambda \to 0$, and the answer I get is $(x-y)^{-2}$. Then the limit as $\lambda \to \infty$ is just 0. The theory you would obtain for the first case is the CFT of central charge one, the massless free boson. That's what $\lambda \to 0$ does, kills the mass. The other case, at ∞ is the trivial CFT with central charge zero and the Hilbert space is just $\mathbb{C}$.

This happens a lot, the infrared is trivial and the ultraviolet is more interesting. The Hilbert space in this example is $L^2(\mathbb{R}) \otimes \mathscr{F}$ (the Fock space $\mathscr{F}(L^2(\mathbb{R}))$).

Let's talk about the "moduli space of 2 dimensional quantum field theories. When you fix a quantum field theory, it comes in a family, so these come with flows. Then I might have fixed points and fixed submanifolds, so these are the conformal field theories. One would expect that, it's more of an idea than anything rigorous. These are the sources and sinks of these flows.

The conformal field theories have more symmetries, conformal symmetries, not just rotations and rescalings. So there are certain correlators that one can compute. One can try to explore by perturbation theory a neighborhood of the theories that can be computed exactly. This allows you to say something about a neighborhood of each of these conformal field theories. There are few lines where you can say something about theories far from a conformal field theory. These are called integrable. You have an infinite number of conserved quantities also away from the endpoints of the flow. There are some lines where there are an infinite number of conserved charges. That's one motivation for looking at conformal field theory. Quantum field theory at long and short range is described by conformal field theory. That's one relation to physics.

An example of the integrable world is $cos\phi$ which is called [unintelligible].

Now I would want to go to lattice models. In fact I want to discuss just one, the mother of all lattice models, the Ising model. It can be defined in any dimension, I am doing it in two only. It can be solved exactly in one and two dimensions, if you can do it in three dimensions you're famous. The lattice Λ_L are pairs $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ so that $|m|, |n| < L$. Then a spin configuration is a function $\sigma : \Lambda_L \to \{\pm 1\}$. To each spin configuration I assign an energy

$$E[\sigma] = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

where $\langle i, j \rangle$ mean neighboring sites, where only one of the lattice variables is changed, by exactly one.

This lattice model will not be a topological lattice model, which depends on the precision of the lattice and the spin configuration and the ground state. Never mind, it's still interesting. Next you define the partition function which depends on L and β the inverse temperature,

$$Z_L(\beta) = \sum_{\sigma} e^{-\beta E[\sigma]}$$

This has the advantage of being finite and well defined. You want to take the limit for L going to infinity. This won't make sense for the partition function but it will make sense for the analogue of the correlator.

The correlator of two spins

$$\langle \sigma_i \sigma_j \rangle_{(L)} = \frac{1}{Z_L(\beta)} \sum_{\sigma} \sigma_i \sigma_j e^{-\beta E[\sigma]}$$

for i and j in Λ_L. I can say $\langle \sigma_i \sigma_j \rangle = \lim_{L \to \infty} \langle \sigma_i \sigma_j \rangle_{(L)}$

The calculation of these things is very difficult. One thing that one can figure out exactly is $\langle \sigma_{(0,0)}, \sigma(n,n) \rangle$. Let me say that $s = \sinh(2\beta)$. Then this correlator is

$$\begin{cases} s < 1 & : \quad (\pi N)^{-1/2} \frac{s^{2N}}{(1-s^4)^{1/4}} (1 + O(N^{-1})) \\ s = 1 & : \quad (const) N^{-1/4} (1 + O(N^{-2})) \\ s > 1 & : \quad (1 - s^{-4})^{1/4} (1 + O(N^{-}2s^{-4N})) \end{cases}$$

This is the first time we see phase changes. So at $s > 1$ taking these apart they go to a constant, there's long range order in my system. At $s < 1$, high temperature, the correlation decreases exponentially as I go to long distances. The interesting point is between. Here my correlators decay with a power law of the distance. That is where my conformal field theory is interesting.

Can I take the continuum limit? I could mean two things, adding more lattice points or take my correlator and measure the correlation further and further apart. Let me define

$$\langle \sigma(x_1) \cdots \sigma(x_n) \rangle = \lim_{\lambda \to \infty} \lambda^{\Delta n} \langle \sigma_{[\lambda x_1]} \cdots \sigma_{[\lambda x_n]} \rangle$$

taking the lattice points closest to λx_i. I need a compensating Δ to prevent this from being zero. We do experiments on spins that are macroscopically separated, so now we calculate this, or try to, with conformal field theory.

Clearly, with this definition, this should be scale invariant afterward. This will be, for $\mu > 0$,

$$\langle \sigma(\mu x_1) \cdots \sigma(\mu x_n) \rangle = \mu^{-n\Delta} \langle \sigma(x_1) \cdots \sigma(x_n) \rangle$$

So Δ will have to be $\frac{1}{8}$ (called anomalous dimension) to cancel the $\frac{1}{4}$ in the case we've done, with the two point function. The continum limit at $\sinh(2\beta_c) = 1$ which has the explicit solution $\beta_c = \log \sqrt{1 + \sqrt{2}}$. Many different lattice models will give rise to the same critical behavior. If you put the Ising model on the triangular lattice, numerically you get the same $\frac{1}{8}$ and the same correlators.

You may wonder about these correlators, you may want a functor. So you want a functor, say, from Ising, take the category of 2-complexes where the objects are closed 1-complexes and the morphisms are 2-complexes which may have a boundary, and then I have a map from the boundary of one piece to the other.

So if $H_S = \mathbb{C}^2$, and to B we apply $H_S^{\otimes |B|}$, and then $C(\Lambda : B \to B')$ assigns to v in $C(B)^* \otimes C(B')$ "the sum of configurations with boundary conditions v." You can't take a continuum limit, because you don't know how to refine, say, a thing on the pair of pants. So there one needs to refine what one means by the continuum limit.