Operads

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1 Kevin W.

[I do not take notes on slide talks]

2 Bruno Vallette
Homotopy and deformation theory of algebraic structures

Thank you for the invitation. The idea here is to study at the same time algebra, homotopy theory, and at the same time quantum field theory or CFT. Between the first two they do not mix very well, so there are nice things that appear. Then there is a homotopy theory of QFT, and also there are algebraic structures in physics, and the common point in this discussion is operads or props. The experience I have had with mathematicians, they say they don't want to hear if they hear a new word. A physicist, when he hears a new name, he wants to hear more and more about it. For a physicist he has vertex algebra, BV algebra, Feynmann diagrams, for the mathematician, homotopy theory, algebra, and so on.

So now to begin. Let's say we have a category of algebraic structure, for example, my toy model, associative algebras. These don't have nice homotopy properties. The idea is to model this category by one object, an Operad. A capital O will denote actual Operads, lowercase will include PROPs or properads or whatever.

So let's introduce cofibrant, or projective, or quasifree resolution (replacement) Operad. In the toy example we get the category of algebraic structures up to homotopy, so these are homotopy algebras, and we'll take the A_∞ resolution from Stasheff, and so we have A_∞ algebras. So the category of associative algebras sits inside the category of A_∞ algebras. I can do basically three things here. You can develop homotopy theory for the resolved (homotopy) algebras, you can solve the transfer of structure, and finally you can study the deformation theory of algebraic structure. For these three, I will describe cobar bar, the transfer theory has been done by Kadeshvilii, Merkulov, Kontsevich, Soibelmann, and the deformation theory is straight out of Gerstenhaber.

[Aside: If you have two spaces which are homotopy equivalent, now the idea is, if you have an associative algebra on Y, you would like to transfer it to X. You can do it in a naive way. An associative algebra is a dg module and then you have $\mu : A \otimes A \to A$ which should be associative, so that $\mu(\mu(a,b),c) = \mu(a,\mu(b,c))$ for all a,b,c. So I can write this as a tree. With this picture, how can I define μ on X? It has to eat two elements of X. You go to Y, you can do the product, you can go back, and I will check the associativity of this product, and you have the following tree: [picture], you would like that to be [picture].]

So now the paradigm, the toy model, is associative algebras and A_∞ algebras. Okay, so this is the definition of an associative algebra. Here comes the point, you want to do algebra and homotopy theory at the same time, this category is too small for the homotopy theory to hold, so what is an A_∞ algebra?

So I will work with a field of characteristic zero and the underlying category of dg modules. So what is an A_∞ algebra on a dg-module (A, d_A) is a collection of operations $\{\mu_n : A^{\otimes n} \to A\}_{n\geq 2}$ such that $\delta\mu_n = d_A(\mu_n) \pm \mu_n(d_{A^{\otimes n}}) = \sum_{k+\ell=n+1} \mu_\ell \circ \mu_k$, and I will say $|\mu_n| = n-2$ Take this as a definition. Now this is a binary product and a bunch of other things. Why does this deserve the name of A_∞ algebra?

The first equation says that d_A is a derivation with respect to μ_2. In $n = 3$ on the left hand side you have $[d_A, \mu_3]$ which is the sum of all the trees with two vertices so that the number is three, so you have the associator of μ_2. So it's not associative, but it's associative up to homotopy.

Then μ_3 and μ_2 satisfy another relation up to homotopy, and so you need μ_4. You could have some other definition. [If you have the transport theorem, any other one would be equivalent.] This is unique up to isomorphism, but if you are looking for other resolutions, we have them. If you ask for the same kind of relations, the answer is no. The quasifree resolutions are unique up to quasiiso.

Now, what do you want to have? Well, we can do homotopy theory for A_∞ algebras. If you go back to the example of spaces, we want to know when maps are homotopic, so we need to know the notion of maps. What is an A_∞ algebra morphism $A \to B$? You ask for maps $f_n : A^{\otimes n} \to B$ and such that a certain relation. This is unique in a certain sense. These can be composed, why I will tell you later. You can define the notion of morphisms between the two, and you can compare them, and there is the notion of A_∞ homotopy. We can compare these. How do we do this in a more or less explicit way. Now I need to develop some machinery. Let me recall the bar and cobar construction, which go back to Eilenberg-MacLane and Adams. You have the category of dg associative algebras, and then the category of dg coassociative coalgebras. These are related by the functors Ω and B, cobar and bar. How do people define the bar construction for an associative algebra? Start from A and construct the cofree coassociative algebra $T^c(sA)$ on the suspension of A. The underlying space is a tensor module, and the Δ is the deconcatenation map. We want the differential to be a coderivation, so there is a unique coderivation which extends, which is characterized by its image on the generators $T^c(sA) \to sA$. So we want to project on $(sA)^{\otimes 2} \cong s^2 \otimes A^{\otimes 2} \stackrel{s^{-1}\otimes\mu}{\to} sA$.

Somehow we did something very particular. What is the shape, sorry, I forgot to say, this derivation squares to zero because A is associative. So in general, what is a square zero coderivation on $T^c(A)$. A coderivation on this cofree coalgebra is characterized by its image on the cogenerators, which means that for $(s \otimes A)^{\otimes n} \stackrel{d_n}{\to} s \otimes A$ for $n \geq 1$. If d has degree -1 then all of these have degree -1. This is a set of maps $\mu_n : A^{\otimes n \to A}$ of degree $n-2$. What does it mean that $d^2 = 0$, which is true if and only if there is an A_∞ algebra structure on A.

At the end of the third talk I'll have four equivalent definitions. So here you see that we had the bar construction from associative algebras to coalgebras. This extends to the category of A_∞ algebras. These then have something to do with coassociative coalgebras. I can now give a reason or alternate definition for A_∞ morphisms. This is a morphism of dg-coassociative coalgebras $B_\infty A \to B_\infty B$. With this definition you see that they can be composed. Then A_∞ algebras with this definition of morphisms form a category.

What is the purpose of this category. If you consider associative algebras, and you want to do homotopy theory for them, you can put weak equivalences to be quasiisomorphisms, fibrations to be epimorphisms, and you can prove that this puts a model category structure on dgAs. What's the purpose? When we do this homotopy theory, we want to understand the homotopy category, $H_0(ass)$, and this is equivalent to the category of $(ass, A_\infty$ morphisms). You use the homotopy relation between coalgebras. As Dennis said, these seem onely like this coalgebra here, you have here cofree colgebras which live in dg coalgebras. What is an A_∞ algebra? A square zero derivation. A homotopy there defines the homotopy there. [Handwaving and rapid talk]

So far we have understood the homotopy category on A_∞ algebras.

So a model category structure, you have three special classes of morphisms, weak equivalences, fibrations, and cofibrations satisfying several things. It allows you to study homotopy theory.

The bar construction preserves quasiisomorphisms. The cobar construction does not preserve quasiisomorphisms, only between two connected coalgebras. You would like to define weak equivalences so that we form a Quillen adjunction. You define instead weak equivalences for coalgebras to be a map whose image under Ω gives a quasiisomorphism. Then weak equivalences are quasiisomorphisms, but there are many more weak equivalences. Taking this definition, you can show you have a model category structure on dg coassociative coalgebras. This lets you understand the homotopy theory. A_∞ algebras with weak maps, this is due to L-H ([unintelligible]) which says that A_∞ algebras are fibrant objects in the model category of dg coalgebras.

So now I have two things to do, the transfer theorem and then deformation theory. What is the transfer theorem. Let me start (it exists by abstract nonsense). Let me say I have R a strong deformation retract of A with h, i, π, then it's very easy to transfer. We'll do the same naive thing at the beginning. I need to define maps $R^{\otimes n} \to R$. So Kontsevich and Soibelmann said, take the sum over all planar trees with n inputs, first move into A with i, and now perform the indicated operations, and then you come back. If you check the degree of this map, it doesn't work, and so you label the internal edges by h. As an exercise, this gives an A_∞ algebra structure on R.

[If a homotopy is like a deformation, and if you use a "propagator," this looks like the Feynmann graph formalism].

So yes, and why this formula? One remark, it's possible to transfer this here, and these maps extend to an A_∞ morphism, and the two algebras are really equivalent. So in particular, R might be the homology of A. Working over the field, you can do this, and so any associative algebra transfers to the homology, and there defines Massey products. If this is an associative or A_∞ structure, then you get the Massey products.

Why to we want these Massey products? You may say, here, on the left hand side you have a chain complex where $d = 0$, as Gabriel said. Because of this, μ_2 is strictly associative. So if I take the formula I take at the very beginning, it works, why do I want to bring the whole structure? This is in order to be able to reconstruct A in the homotopy category. The higher products are to be able to contain the information, the homotopy class of A. You start from $H_.(A)$ and, you take $\Omega B H_. A$, which is a dg algebra, this is called rectification. You have quasiisos from $H_. A$ to both this and A, and so these are the same class in the homotopy category. That's one application of the Massey products, and you can do the same thing with any Kozsul operad.

I finish with the deformation theory. The idea is to be able to represent the set of A_∞ algebra structures with solutions to a Maurer Cartan equation. So let A be a dg module. Consider

$$\mathfrak{g} := \prod_{n\geq 2} Hom((s\otimes A)^{\otimes n}, s\otimes A)$$

This is the product, not the sum. So you take the star product $f \star g = \sum f \circ_i g$, which is preLie, meaning that the associator is right symmetric, so that $(f \star g) \star h - f \star (g \star h)$, so that when you take its bracket you get a Lie bracket. What is the Maurer Cartan equation there? I can write $\partial\alpha + \alpha \star \alpha = \partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0$. So the proposition, or exercise, since you have an hour break, is that solutions in $\mathfrak{g}_A$, correspond to A_∞ algebra structures. This gives a third equivalent definition of such structures, and now you can define deformation theory for such algebras.

[Equivalence]

I will say that they are homotopy equivalent if there are two A_∞ morphisms whose composition are homotopic to the identity.

Okay, so, we can twist, consider $\mathfrak{g}_A^\alpha$, where everything is the same except the differential: $(\mathfrak{g}_A, [\,,\,], \partial + [\alpha, \quad])$. Then the homology of this is the (contains the) obstruction to deform the A_∞ structure α. An honest algebra, associative algebra, this gives Hochschild homology, but here you get everything at the same time.

3 Bruno II

So far I did only what is in green on the board. To extend this I need a new tool, what do we need? We need the notion of Operad, to algebraically model what happens on a certain type of model. Manin in mathematics in metaphor, he said that this was the revolution in algebra in the 20th century. The notion came from homotopy theory. In the fifties and sixties, there were a lot of homotopies appearing. They needed a way to organize these. So now the definition. I'm going to draw my favorite table. We would like to model certain types of relations. We'd like to compose them. With one input and output, you do it in a ladder. Then you want many inputs, and then many of both. To model this I will need a monoidal category, and in that a monoid. Finally we need modules over the monoid.

operations	$1:1$	$n:1$
composition	$\circ$	gluing in a two level tree
monoidal category	$(Vect, \otimes)$	($\mathbb{S}$-mod,)
monoid	algebra	operad $P \circ P \to P$, effective compositions
modules	associative algebra	Steenrod algebras
$U(\mathfrak{g})$		
$k[\Delta]/\Delta^2$	Frobenius, involutive Lie bialgebra, TCFTs	

For $1:1$ you have $Hom(A,A)$ and you compose them with concatenation. This happens with vector spaces, and then you have a map $P \otimes P \to P$ which is associative. a module is an morphism of associative algebras $P \to Hom(A,A)$, so a structure is a map. You want something that models all the different types of algebras, like associative, Lie, Gerstenhaber, so on. So I want $End(A)$, the endmorphisms from $A^{\otimes n} \to A$. I compose these in a tree, like take three maps f_1, f_2, f_3 and plug into a 3 to 1 map g. The idea, we have operations, and there are several things we can do. We can permute the inputs as well. We want to consider an $\mathbb{S}$-module, which is a collection $\{P(n)\}_{n>0}$ where this is a right $\mathbb{S}_n$ module. You can act by $\mathbb{S}_n$ on $End_n(A)$. So $P(n)$ stands for the operations with n inputs. So what's the monoidal structure? We have $P \circ Q(n)$ which is

$$\bigoplus_{k \leq n} P(k) \otimes Q(i_1) \otimes \cdots Q(i_k)$$

So first I have to induce a representation from $\bigotimes_{S_{i_1} \times \cdots \times S_{i_k}} \urcorner[\mathbb{S}_n]$ and then I want to take the coinvariants. What is the relation to check for associativity, you have to check that if you take three levels, it doesn't matter which order to do it on. Then we pass to the monoid, where we need two maps. The associativity of the map $P \circ P \to P$ encodes the relations satisfied by elements of P. I will give examples.

Two remarks. First, you have to be very careful. The first column is operations with one input, this is a particular example of the second. There are two relationships between associative algebras and operads. An associative algebra is an operad. But there is one operad which models the category of associative algebras. There is one operad for which modules over it are associative algebras.

The definition, what is an algebra over an operad. A P-algebra structure on A is a morphism of operads $P \to End(A)$. It means that on P you have a vector space which encodes operations. For example, let P have $\urcorner$ in arity n; in this example, forget the symmetric group. What is my space of formal operations? I have one operation for any n. Let me call that Ass. What does that mean? First, I have a concrete map $m_n : A^{\otimes n} \to A$. It's a morphism of operads. In Ass, the composition takes two levels of operations and is supposed to give one operation. Everything is one dimensional and so I have no choice. This is really the naive composition. Check that it's associative. So this forms an operad. Now I'd like to understand algebras over it. I can either do the composition inside P or I can look in my algebra A. What does it mean? Let's start with μ_2. Then $\mu_2(\mu_2(,),)$ should give μ_3 But if we start from the other guy $\mu_2(,\mu_2(,)$ it gives the same thing, which is also μ_3, so μ_2 is associative. So here Ass algebras is exactly the category of associative algebras. How many operations from $A^{\otimes n} \to A$ do you have? Forget the symmetry and you have just one. So this map gives you the shape of the relations.

So [unintelligible]defined PROPs. Later May and Bordmann-Vogt introduced Operads which were easier. But now we will move to props. Let's now go $A^{\otimes n} \to A^{\otimes m}$. It would be good to model Frobenius bialgebras, involutive Lie bialgebras, TCFTs, also many other things. I will try to be as lazy as I can. I don't need the full structure of a PROP. I want n inputs and m outputs. To be lazy, I'll be looking only at connected parts because in most of the examples this is freely generated on the connected part.

So we need this to be a bimodule, with a left $\mathbb{S}_m$ action and a strictly commuting $\mathbb{S}_n$ right action. I need to put more things for the monoidal structure $\boxtimes$, but we don't need to see the details. You just need to consider three-level connected graphs. Every operad thing is a particular example of the properad thing.

So a properad is a monoid in a monoidal category. You have two compositions in a PROP. It's a "$2 - monoid''$. So homotopy theory is much easier for properads than for PROPs. For PROPs there is no bar or cobar, no Koszul duality. For conformal field theory you only need connected parts, and for many examples you don't need this. You generate the free horizontal product on a properad. Since it is freely generated, if you take the free PROP algebra generated by an algebra.

A P-algebra structure on A is a properad morphism $P \to End(A)$. Let me call $Frob$ the following. For any n, m, I consider the n, m corolla marked by g, for $g > 0$, for $n, m \geq 1$. I have to define the composition. It's the trivial group action on both sides. This is supposed to model a commutative operation. I mark the vertex by a nonnegative integer. When I compose two of them, say, n to m marked by g and q to p by h, gluing along k edges, I get $n + q - k$ to $m + p - k$ marked with $g + h + k - 1$, the most naive thing. So you do the same thing, and a Frob algebra is a commutative Frobenius algebra. This has a binary product and a binary coproduct on A, commutative and cocommutative, associative and coassociative. Then you ask for the module relations usual for Frobenius algebras. The other way around, you take a module with these, and allow all possible compositions, you get the guys from $Frob(m, n)$. Algebras here are commutative Frobenius algebras, with no traces and so on.

Now two remarks. First, you can go a little bit further, I have operads and properads for algebras and bialgebras. If you want to model the action of something with several spaces $A_1, \ldots, A_k$, you add colors on the input and output, which will lead to the notion of colored operad. Operad with a capital O is all of these different types [Ed: I typo'ed this earlier]

If the algebra is finite dimensional, then you have a trace. How do you encode that? Then you add wheels, which are contraction operators. I encountered that in a very regular way. The computer scientists have confluence, which corresponds to Kozsuality. Now I take five or ten minutes to say what I will do with this.

Now we paid the price to do this, the idea is now to do homotopy theory on the level of Operads. Once again we'd like to understand the homotopy category of properads themselves. Now dg properads have a model category structure where weak equivalences are quasiisomorphisms and fibrations are epimorphisms. So then we can do homotopy theory on this level. So this implies that transfer works for cofibrant properads. There is no formula here. We know by abstract nonsense,

Proposition 1 *Cofibrant properads are retracts of quasifree properads. Over characteristic zero, any properad admits a quasifree (cofibrant) resolution.*

Now I'll finish by making explicit a quasifree resolution. I have to say what this word means. So, now I will do it. A quasifree Operad has as its underlying space a free Operad on some generators, and then a square zero derivation.

What is a free algebra? You put these guys one next to each other. The free operad, you do the same thing, pile them on top of each other. Then you formally compose everything in connected graphs with labels on the inputs and outputs, with elements labeled everywhere, so it's just a connected graph, take the sum over all connected graphs with flows and no cycles. You compose with gluing. This satisfies the universal property of a free object. A derivation satisfies, d of the composite of two operations, is the product of d on each one. If you only allow graphs with one input and one output, you get the Liebnitz relation for an algebra. So this is determined by its action on the generators. So on $F(X)$ the data of d is equivalent to a map $X \to F(X)$, so a sum of graphs. Now I can split this with respect to a number of vertices, so I have $d_1, \ldots, d_n$, which is maps $d_n : X \to F(X)^{(n)}$, graphs with n vertices. The final point is, $d^2 = 0$ means that these maps satisfy a structure. If you do the sum $\sum d_k \circ d_\ell$ you get zero. I will stop here? It means that you have a square zero derivation, these satisfy a certain relation. This is the kind of resolution you're looking for, it says we have a homotopy coproperad. When all of these are zero except d_2 this is the Kozsul case. Cofibrant resolution are retracts of quasifree. Any quasifree is too hard, so for now we will consider only $n = 2$. Thank you.