Operads

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1 Paul F.

As you can tell by the outline, I'm a physicist. I don't know my audience.

1. Anyons, anyone know what they are?
2. RCFT, R is for rational.
3. Quantum Hall Effect, you probably don't know that.
4. Edge mode, that's a consequence of number three.
5. One and four and TQFT

We think of a physical system with a space of states which are a Hilbert space. Then a state is called a quasiparticle or a particle. If I give a label, that's the type of particle. There's a notion of statistics of a particle, which tells you the behavior of the state, the wave function under exchange.

Say we have two particles at positions x_1 and x_2, then the wave function $\psi(x_1, x_2)$ gives the probability amplitude that they are at these positions, such that $\int_R |\psi(x_1, x_2)|^2 dx$ is the probability that the two particles are in this region.

To summarize all of quantum mechanics, the weirdnesses come from the probabilities being squared.

Now we can ask what happens to that wave function when particles exchange. So basically I move a particle around the other, let's move them around so that they never come close to each other, so what happens to the behavior of the wave function under this process. So this says, how is $\psi_{ab}(x_1, x_2)$ related to $\psi_{ab}(x_2, x_1)$. In three dimensions, there's a simple classification, because in three-space nothing should change dramatic under slow adiabatic, well, small variations. Then you get in three dimensions, now notice if I do this process twice, if I bring a around and then bring a around again, then this process, because I have a third dimension out of the board, I can deform this to doing nothing. This says that two exchanges gives you the same wave function back again, **1** acting on the wave function. Therefore, $\psi(x_1, x_2) = \pm\psi(x_2, x_1)$, assuming $a = b$. The plus or minus is that it's a $+$ if we're dealing with bosons, and $-$ if you're dealing with fermions. There's a much cooler idea, in two dimensions, you can't deform, because going twice around, you can't deform this without bringing the particles through each other. One example, if you have $\psi_{aa}(x_1, x_2)$, you slowly take these arguments around each other, when you come back you get a phase $e^{i\alpha_{aa}}$ times $\psi_{aa}(x_1, x_2)$. This is for a half path, one exchange. The path gives you the behavior of this function. This is a multivalued function.

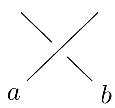
[Here's something that may clarify. Why, when you look in the mirror, does it exchange left and right but not top and bottom. You have to have a symmetry to ask whether the wave functions are related.]

When $\alpha \neq 0$ or π, we call such a particle an anyon. So any experiment should be the same for all time and space, for a given type of particle. This is a joke because they have "any" statistics. These things have been observed.

Now I'm going to tell you something about the structure that has not yet been observed. This is what people in the racket call non-Abelian anyons. There's a paper where people tried to call them non-Abelions. Some people suggested to name them Morons after one of the authors, but his collaborator didn't like it.

What happens when you have different as and bs here. If you have a degenerate (space of states), two states with different types of particles which have the same energy. It's an eigenspace of the Hamiltonian of dimension greater than one. When you do this exchange, you're not changing your energy. You give it a slight little nudge and it goes around. If you have multiple states, you can move around within this eigenspace (of the energy operator or Hamiltonian) just by this exchange process. And so, let me write down something more concrete.

You've got some ψ_{ab} and then you've got, let me label that just by a single label, c, and then under this process, you get $\sum_d M_{cd}\psi_d(x_2, x_1)$. So if you want to list all the possible two particle states then this is multiplying by some matrix. This matrix has a name which hopefully will be familiar to some of you, this is called a braid matrix. I should call it B, not M, and you can think of this pictorial,



and the particles at the end might be different than a and b.

So if I put two fermions together, if I bring one around the other, I pick up four minus signs, so you pick up $(-1)^4$, so a composite of two fermions is a boson. The same kind of thing happens in the non-Abelian case. If you have a and b then they act together, say, as c, a bound state of anyons a and b. But what c is can depend on how you put them together, it's not necessarily unique. If you have a degenerate eigenspace, you might end up with different cs. It's a set of vertices depending on how you build things up.

If a and b were spin 1/2 particles, then you might imagine that putting them together you might get spin 0 and might get 1.

When you do the braiding (changing the name of exchange), you can change what label you end up with. [degeneration of discussion]. To summarize, we have two key ideas. We've got some eigenspace of states and there are two physical properties that we wonder about. One is what happens when we braid things together. If we have, well, we have four particles a through d, and then ask what happens, a and b form some state e and we demand c and d also form e. Then we do this braiding, and the braid matrix B tells us what happens when you take this to a set of states a with c and b with d with f as the state that is output. The other notion we have is fusion, let me skip, well, we have a and b bound in state c. That summarizes the anyon part. Let me say something about CFT.

What does this have to do with conformal field theory? It sounds like a completely different story. To say some physics words, what happens when you have a field theory in two spacetime dimensions which describes critical systems, and what critical means is that there is no length scale involved. That sounds utterly different. I'm just going to give a quick explanation of why what I wrote before has to do with this study. What it is is that when you have $2d$, this is special to physicists, you can write things in complex coordinates z and $\bar{z}$ instead of x and y, and what happens is that correlation functions will decompose into the sum over the products of a bunch of things that depend on z and the ones that depend on $\bar{z}$, $\sum c_j \langle \phi(z_1) \cdots \phi(z_n) \rangle \langle \phi(\bar{z}_1, \ldots, \bar{z}_n \rangle)$. What is the probability that the field will be at this strength here given that it is at this other strength there. You can ask these things with lots of objects. The experiments you do, the numbers you get can be related, and you can derive differential equations, and you have a correlation function which is a function of $\bar{z}$ and another that is a function of z. Then you can reduce this to a computation of those things.

Let me write this more succinctly as a correlation function $\sum c_{jk} f_j(z_1, \ldots, z_n) g_k(\bar{z}_1, \ldots \bar{z}_n)$. There's fantastic stuff going back to the 80s on how to compute these things. You might, well, the picture I drew not that long ago looked like this. What that was, I said a through e were anyonic labels. We braid and we get this picture. A conformal field theory, you can think of this as being not so much a particle but the value of a field at a point. You have different types of fields. So the object we're computing here has particle labels and a position label and you'll get a number out. That's starting to sound like the same thing, we have a bunch of labels and it spits a number out. How does that function depend if we change the positions around. We can ask for the monodromy of these objects. On physical grounds we expect that, imagine a sphere with the electric field all over it, with four points on it, and we compute a probability, as we go around again, what happens to the probability, the whole thing shouldn't change, but the bits in the decomposition can change as long as they change in the same (opposite) way. So you can pick up phases in conjugate pairs which cancel. With different labels, these could be multiplied by a matrix and by the inverse matrix.

When people learned how to do conformal field theory, it was recognized right away that there were interesting properties from braiding. This got abstracted more and more and people realized with, Moore and Seiberg were the culmination, that you can characterize each RCFT by a set of braiding and fusing matrices which satisfy consistency conditions. The way to derive these matrices is to impose consistency conditions on braiding, so one of them is the Yang-Baxter equation. The remarkable thing about conformal field theory is that the braid matrices satisfy these consistency conditions. So you get that conformal field theory satisfies this. There's also the pentagon equation. I promise not to bring it up again with a study of exact S matrices and integrable models, where it's the bootstrap relation. If you fuse, braid, and unfuse, that's the same as braiding before you fuse with both prefused particles.

Now go back to the first forty minutes, the anyons where wave functions changed by moving them around. The process of exchange was the same. We can think of the braid and fusion matrices themselves and you get the same type of thing, the same braiding (and fusing). In the Hall effect, which I'll get to next, both of these will happen in the same system, and these will come together in a physical correspondence.

2 Mike Freedman

Dennis told me once that a topologist was a person who was born knowing what functoriality was. That's a good working definition. Topologists learn physics by starting there, learn what a TQFT is, then probe a little deeper and find that there are CFTs, and now there are vertex algebras. If you dive a little deeper you get to QFT, this is what physicists talk about, real distance. You can keep going to FT and QM. If you really want to understand quantum mechanics, you have to understand chemistry, you're desperate, you get into devices, CMOS. Finally, if you proceed far enough you're learning about Bessel functions and solving differential equations. Of course physicists do this in the opposite direction, so they know what they're talking about. For me this was a humbling personal experience to move down from the top of the tree. It was easier to think in an ambient manifold. It was humbling to deal with the things I had dismissed earlier. If you step into a graduate course in QFT, to start in the middle, the mathematicians complain how ill-defined and poorly organized the subject is. On a superficial level you run into divergences. A Feynmann diagram needs to be evaluated and at the least you get log divergence. At a finer and finer scale you get ultraviolet divergences. At longer wavelengths you get infrared divergences. These annoyances, plus that it's not clear what the Hilbert space is, well, it can be pretty frustrating. The physicists have answers to these things. T'hooft handles logarithmic divergence with dimension regularization, $4 - \epsilon$ as the dimension. The ultraviolet are dealt with using a cutoff. An infrared divergence implies that you've asked the wrong question. I wanted to give just one concrete example.

I would say that roughly fifty percent of quantum field theory is Gaussian integration, perturbed. It's a mistake to try to learn field theory without integration tricks. All of the intuition comes from bringing variables out of the integral, out in front of the exponential.

The first perturbed Gaussian integral is

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2} - gx^4}$$

where g is a real number greater than zero, often written $\lambda/4!$. You can break this up and you write this as a summation

$$\sum_{n=0}^{\infty} -g^n \int \frac{x^{4n}}{n!} e^{-\frac{x^2}{2}}$$

Now we have many problems, to find higher moments of a Gaussian. There are many tricks, involving differentiating with a source, and then setting sources equal to zero. Things have to tie together and that's what leads to Wicks theorem. The answer is

$$\sum_{n=0}^{\infty} frac(-g)^n (4n-1)!!n!$$

Now !! is interesting, physicists don't believe in recursive functions, they know there will never be a factorial of a factorial. Physicists believe that big numbers couldn't come out of small numbers, so they think there has to be a reason for separation of scale, like gravity's relative weakness.

The right hand side (that was a digression) is a perturbative expansion of the left hand side. This means it tells you about the behavior as g goes to zero. It isn't telling you anything about the radius of convergence, which is obviously zero. I said g were positive, but if it became negative and then [unintelligible]would have to dominate.

I want to explain that this is a very good preturbative expansion even though it doesn't converge. g is the fine structure constant in QED, so it's like $\frac{1}{137}$. The combinatorial part is superexponential. The magnitude decays for a while and then expands. I actually did the calculus and the arithmetic, and you can figure out how long they shrink for. They shrink for the first nine and then blow up to around unity at 40. The error is always less than the size of the next bump. When you put in the numbers this gives an error of roughly 10^{-5}. For a real value of the parameter, if you know when to stop adding, you'll get five decimals of accuracy.

This is the most naive way. There are many more sophisticated tricks. You don't have to stop analytically, you can probably get ten or twenty decimal places.

I apologize if you did take a course in quantum field theory. There's lots of room for mathematicians to work, and we have finite lives. I want to relax about the analytic issues. I want to look at this from a set theoretic point of view. I've been playing with a generalization of field theory.

So many people here do a mathematics with a 2 in front of it, like 2-category, well I'm going to do a 2-field theory. What's elevated is the level of "functional." Now functional is an old fashioned word for functions on functions. This is a game for me, but serves a certain very useful purpose in my own mind, it clarifies where field theory starts and ends.

What is a field, and a 2-field?

A field is a function from $\mathbb{R}^4$ to somewhere, let's say $\phi \in R^{\mathbb{R}^4}$. The place of $\mathbb{R}^4$ would be taken by $(\mathbb{R}^4 \times CY)^{S^1}$ in string theory.

A 2-field will be something like $\| \!\!\!\!\circ$ [oops. Got lost making that symbol.]