Stringy Topology Notes
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One reminder, you need to, this is an MSRI rule, you need to sign in with our secretary to get reimbursement.

It is my pleasure to present professor Stolz.

1 Stolz, Generalized homology and quantum field theory

This is the third part of the talk. Our work is really based in field theory a la Atiyah-Segal, but on top of that we put in a degree, in CFTs the extended structure of going a step down to zero dimensional manifolds, and the supersymmetry, which it is my job to explain.

I want to show how there's a natural modular form attached to a supersymmetric field theory.

Roughly speaking, a supersymmetric field theory, a field theory is a functor from the category of bordisms to $Vect$ and in a supersymmetric theory you replace the manifolds with supermanifolds

1.1 Supermanifolds

Definition 1 *A supermanifold of dimension $p|q$ (bosonic/Fermionic dimension) is $X = (X_{red}, \mathscr{O}_X)$ where X_{red} is a space (p-dimensional manifold) and $\mathscr{O}_X$ is a sheaf of $\mathbb{Z}/2$-graded algebras which locally looks like $(\mathbb{R}^p, \mathscr{O}(U) = C^\infty(U) \otimes \wedge[\theta_1, \ldots, \theta_q]$.*

The mother of all supermanifolds is this local model $\mathbb{R}^{p|q}$

Here the subscript red stands for reduced.

[That can't be the definition, you need it to be locally locked in.]

I'm identifying the sheaf with the functions on $\mathbb{R}^p$, these are isomorphic as local rings.

[You have to check a little lemma.]

[That's not a lemma, that's a definition.]

[[unintelligible]a [unintelligible]definition]

Okay, so let's, if I have a supermanifold $C^\infty(X) = \Gamma(\mathscr{O}_x)$ a $\mathbb{Z}/2$-graded algebra.

[Why does one look at these things?]

Well, for us,

[Not for you, in general.]

Physicists want to look at Fermions, which have different properties than bosons.

You have a lot of nilpotent things, $C^\infty(X)/\text{nilpotents} = C^\infty(X_{red})$.

What would a vector field be on X? It's defined as a graded derivation on $C^\infty(X)$.

Example 1 $C^\infty(\mathbb{R}^{1|1}) = C^\infty(\mathbb{R}) \otimes \wedge[\theta]$.
Vector fields, for example, an element is $f = g + \theta h$ *where* g *and* h *are normal functions. A derivation could be like* ∂_t *or* ∂_θ*, or a combination like* $Q = \partial_\theta + \theta\partial_t$*.*

So now the only calculation in this talk, $Q(g + \theta h) = h + \theta(\dot{g} + \theta\dot{h}) = h + \dot{g}\theta$*.*

Then $Q^2(g + h\theta) = Q(h + \dot{g}\theta) = \dot{g} + \dot{h}\theta = \partial_t(g + h\theta)$*.*

Unlike the usual vector field, you square this odd vector field and you get an even vector field $Q^2 = \partial_t$*. That's why these are important to us.*

[What does this look like locally?]

It's of dimension $p + q$ just as in a normal manifold but the p is even and the q is odd.

I should finish by talking about maps between supermanifolds. The honest way would be to say it in terms of sheaves. Here we say $\mathscr{SM}(X, Y) = Alg(C^\infty(Y), C^\infty(X))$.

[If I have a bundle over a manifold with a multiplication, is that a supermanifold?]

Yes, every supermanifold is isomorphic to one of these in the [unintelligible]category, but not in the algebraic category.

[For Riemann surfaces what will this be?]

We'll always have one odd dimension.

1.2 Supersymmetric field theories

Definition 2 *(Naive)*
A supersymmetric one dimensional EFT is a functor $E : \mathscr{S}\mathscr{E}\mathscr{B}^1 \to \mathrm{Hilb}$. The objects are supermanifolds Y (closed) of dimension $0|1$ and the morphisms are manifolds Σ of dimension $1|1$ which are bordisms from suprmanifolds $Y_1 \to Y_2$ This has a superEuclidean structure, a local odd vector field up to a sign ambiguity D such that $D^2|_{X_{red}}$ is nowhere vanishing. This gives you an orientation but also a metric by declaring that vector to be a unit vector.

[Sign ambiguity?]

You have $\pm D$ locally. On the Hilbert space side you want the Hilbert space to be $\mathbb{Z}/2$-graded with morphisms trace class operators.

That's a first version.

Definition 3 *A two dimensional supersymmetric conformal field theory is a functor $E : \mathscr{S}\mathscr{C}\mathscr{B} \to \mathrm{Hilb}$, where the objects on the left are manifolds $Y^{1|1}$ with morphisms $\Sigma^{2|1}$ equipped with a superconformal structure.*

What is this? An ordinary conformal structure $T\Sigma^2 \otimes \mathbb{C}$ splits into $T^{1,0}\Sigma + T^{1,0}\Sigma$ where these are spanned by ∂_z and $\partial_{\bar{z}}$.

Here you have $T\Sigma^{2|1} \otimes \mathbb{C} = T^{1,0}\Sigma + T^{0,1}\Sigma + T^{0|1}\Sigma$. In local coordinates this looks like $\partial_z, \partial_{\bar{z}}, D = \partial_\theta - \theta\partial_{\bar{z}}$.

[You really want that global splitting?]

I can find local coordinates, but they are not part of the structure. I have more than just the automatic structure from D. I do get $\partial_{\bar{z}} = -D^2$. I do not have complex conjugation on the big sheaf, I need to get ∂_z by hand.

[This is two complex things at each point? The extra information?]

This is a complex parameter.

[So it's twice as much.]

The remark I want to make, what does this mean, I give weird definitions, if you have a supereuclidean (respectively superconformal) structure on X then you have a Euclidean (conformal) structure on X_{red}, as well as a spin structure on X_{red}.

Now I want to give you the honest definition, working in families instead of with single supermanifolds, like in fiber bundles, I learned this from Dan Freed.

Definition 4 *Fix a supermanifold S. Then I want a functor $\mathscr{S}\mathscr{E}\mathscr{B}^1(S) \to \mathrm{Hilb}(S)$. The objects are bundles of supermanifolds over S, with fiber dimension $0|1$. The morphisms are* *bundles* Σ *with*

$$\begin{tikzcd} Y_1 \arrow[r, hook] \arrow[dr] & \Sigma \arrow[d] & Y_2 \arrow[l, hook'] \arrow[dl] \\ & S & \end{tikzcd}$$

wthich are fiberwise bundle morphisms, then on the other side

$$\begin{tikzcd} \mathscr{H}_1 \arrow[rr] \arrow[dr] & & \mathscr{H}_2 \arrow[dl] \\ & S & \end{tikzcd}$$

A one dimensional supersymmetric Euclidean field theory is $E = \{E(S)\}, E(S) : \mathscr{S}\mathscr{E}\mathscr{B}^1(S) \to \mathrm{Hilb}(S)$ *and compatibility for* $S \to S^1$.

Think about ordinary manifolds, what would this mean? The original functor is smooth. You express the idea that the Hilbert spaces depend smoothly on the manifold you stick in, the operators depend smoothly on the bordisms, and the superness, I don't know another way of expressing this, this is functorial for supermanifold maps.

Similarly I define for susy CFTs.

There are also chiral CFTs, where the operators depend holomorphically on the bordisms. Then you have categories, one for each complex manifold, and you require that the operators depend holomorphically on the holomorphic bordisms.

[You did not mention monoidal categories.]

Yes, I need some other properties that I leave under the rock for this talk.

1.3 Partition functions

Suppose you have a susy 1-dimensional EFT E. Then I associate $Z_E : \mathbb{R}_+ \to \mathbb{C}$ is evaluating on closed manifolds, $t \to E(S^1_t \times \mathbb{R}^{0|1})$ (the circle of length t crossed with $\mathbb{R}^{0|1}$ which has a natural supereuclidean structure.

For a supersymmetric 2-dimensional CFT E I have Z_E from the upper half-space, $H \to \mathbb{C}$ which takes $\tau \mapsto E(T_\tau \times \mathbb{R}^{0|1})$ (the torus generated by $1, \tau$). These are the partition theories.

Theorem 1 *A. If you have in the one dimensional case,* $Z_E(t) \equiv$ *a constant,* $Z_E(t)$ *is an integer.*

B. In the two dimensional case, $Z_E(\tau)$ *is holomorphic and* $Z_E(\tau)$ *expanded in terms of* $q = e^{2\pi i\tau}$ *is* $\sum_{n\geq -N} a_n q^n$ *with* $a_n \in \mathbb{Z}$. *For* $A \in SL_2(\mathbb{Z})$ *we have* $Z_E(A_\tau) = Z_E(\tau)$. *This is a modular function with integral q-form. To be a modular form I need degree.*

C. *If I have a two dimensional susy CFT E of degree n then Z_E is a (weak) modular form of degree n (weight $n/2$).*

Our operators are not holomorphic in the parameters, but our partition function still has holomorphicity, that is surprising and comes from supersymmetry.

Given E a supersymmetric one dimensional EFT we will construct an odd operator D_E. $\mathscr{SEB}^1(pt) \overset{E(pt)}{\to} \mathrm{Hilb}(pt)$, $\mathbb{R}^{0|1} \mapsto H$.

Now I look at $\mathrm{Isom}(\mathbb{R}^{1|1})$, which is a nice superLie group. The structure induces a structure on $\mathbb{R}$ so such a manifold induces an isometry on $\mathbb{R}$. So what sits over $\mathbb{R}_+$, the moduli space of intervals? We have $\pi^{-1}(\mathbb{R}_+) = \mathrm{Isom}_+(\mathbb{R}^{1|1}) = S$.

I want to write down an endomorphism of $\mathbb{R}^{0|1} \times S \in \mathscr{SEB}^1(S)$ where it looks like

$$\begin{tikzcd}
& S \times \mathbb{R}^{1|1} & \\
S \times \mathbb{R}^{0|1} \arrow[ur, hook] \arrow[r, hook] \arrow[dr] & \Sigma \arrow[u, "\subset"] \arrow[d] & S \times \mathbb{R}^{0|1} \arrow[l, hook'] \arrow[ul, hook', "1\times\mu"'] \arrow[dl] \\
& S &
\end{tikzcd}$$

Here $\mu : S \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ action.

[Picture]

So that leads by E to

$$\begin{tikzcd}
S \times H \arrow[rr] \arrow[dr] & & S \times H \arrow[dl] \\
& S &
\end{tikzcd}$$

So $T : S \to \mathscr{T}(H, H)$ takes a smooth homomorphism to a trace class operator. So $Lie(\mathrm{Isom}(\mathbb{R}^{1|1})) \to End(H)$. So this takes a vector field on $\mathbb{R}^{1|1}$ which commutes with D that is, $\langle Q, \partial_t \rangle$, so this maps $Q \mapsto D_E$ and $Q^2 = \partial_t \mapsto D_E^2$.

I will take four more minutes to reap the rewards, the proof of the first two theorems is easy now.

Proof.

$Z_E(t) = E(S^1_t \times \mathbb{R}^{0|1}) = \mathrm{str}(E(I_t \times \mathbb{R}^{0|1})) = \mathrm{str}(e^{-tD_E^2})$, and you get the nice cancellation thing that you probably have seen, and you split into eigenvalues and get

$$\sum_{\lambda \text{ eigenvalue of } D_E^2} e^{-t\lambda} \mathrm{sdim} E_\lambda$$

where $\mathrm{sdim} E_\lambda = \dim E_\lambda^+ - E_\lambda^-$ is zero for $\lambda \neq 0$, so this is $\mathrm{sdim} \ker\ D_E^2$.

For the conformal case you get the annulus A_q and $E(A_q \times \mathbb{R}^{0|1}) = q^{L_0}\bar{q}^{\tau_0}$ ([unintelligible]$(L_0 - \tau_0) \in \mathbb{Z}$) and this is $q^{L_0}\bar{q}^{\bar{G}_0^2}$ for $\bar{G}_0$ odd.

Then

$$\begin{aligned} Z_E(\tau) &= E(T_\tau \times \mathbb{R}^{0|1}) = \mathrm{str} E(A_q \times \mathbb{R}^{0|1}) \\ &= \mathrm{str}(q^{L_0}\bar{q}^{G_0^2}) = \mathrm{str}(q^{L_0}\bar{q}^{G_0^2}|_{\ker\ G_0^2}) \\ &= \mathrm{str}(q^{L_0}\bar{q}^{\tau_0}|_{\ker\ \tau_0}) = \sum_{n\in\mathbb{Z}} q^n \mathrm{sdim} E_n \end{aligned}$$

2 Sullivan

Today I wanted to talk about an internal picture of the category of Riemann surfaces and then I want to use the Thom class I constructed yesterday, the smoothing operator, and use that to construct some operations of string topology. Maybe I'll get to that by the end. Again I want a very elementary discussion of things. Let's start with a fact. Suppose you have a closed Riemannian manifold and some points in it, and you assign some weights, w_1, just weights like this, and total mass is zero, some negative weights, so there's weights here, you don't have to have the same number of points, they, up to scale, an additive constant, there exists a unique harmonic function h, real-valued, mapping from this Riemann manifold, minus these points, that goes to $-\infty$ and ∞ at these points, and the gradient dh has a singularity, if you think of this as a distribution, this satisfies Δh is the measure, the Laplacian. If you look at the gradient in the middle, I realize I forgot the formula, from here to here I forgot the formula, you think of like these are like faucets and these are like drains, and you're putting in and taking out weights, and you look at the volume preserving flow that has minimal L^2 norm and then it will be unique and it will be the gradient.

A second fact is that if $d = 2$ then this unique harmonic function, the gradient is really unique, in $d = 2$ the gradient only depends on the underlying conformal structure. This is only in dimension two, in higher dimensions this depends on the metric.

Maybe I should say a little bit more. The L^2-metric on the 1-forms only depends on the conformal structure. Now we can use this to make a picture, this implies a picture of a Riemann surface with distinguished points, weighted distinguished points, and the picture is, a picture of anything means, this is a geometric picture, so, I mean, a picture of a metric space, if you have a Riemann surface it doesn't have a metric, the negatively curved one of higher genus gives one, but that's a different one. Well, away from the zeros of gradient h choose a scale factor so that $|grad\ h| = 1$ whenever it's nonzero.

Let's put the picture aside, so dh could be thought of, consider something else, you have the Riemann surface with these points, now let's consider holomorphic 1-forms on Σ. As you move around in this moduli space, you get a nice vector bundle, and Riemann surfaces degenerate by pinching curves and this bundle extends nicely over the degeneration. Then you think that you get a Riemann surface with two new points. You have to discuss holomorphic one-forms in the presence of points, so you discuss holomorphic one-forms with first order poles at the points.

There's one condition on residues, that if you take a small circle around any one of these points, the integral around each of these gives the residue, but the sum of these cycles is homologous to zero, so the sum of the residues is zero.

So this is a complex vector space, consider in here the subspace so that all the periods of the 1-forms are pure imaginary. So if you look at the period, a complex number, around anything, it's real part is zero.

Now the one form can be written as something plus i something, the real part is a harmonic one form, all its periods vanish, so it's dh, and the residues, the periods around the small circles, you get real numbers, with sum zero, those are the weights.

In the case of Riemann surfaces, I've embedded this structure into a real subbundle of this complex bundle of holomorphic one-forms with poles of degree at most one and imaginary periods.

I learned from the algebraic geometers, someone who knows algebraic geometry, how this thing works. I want to know what happens as you go out to infinity in the moduli space. Consider the usual picture of moduli space, Thurston started explaining how to study three-manifolds using hyperbolic two-manifolds, so each Riemann surface has a canonical metric, not the one here, the constant curvature -1, these will look like cusps, with things pointing out. Here's a picture of a moduli space near infinity. This picture with a long thin neck between the pieces, that's the picture, the obvious limit of that is to have two things, two disconnected things with cusps. The moduli space of genus two has regions at infinity, some of which are pairs of tori with marked points, this is what happens when you approach infinity in a direction in which a separating curve gets small. Now you can have, that's the only way in genus two to separate. There's only one other piece of the boundary, where you pinch of the nonseparating curve, you get something with two cusps, and then this will give a full compactification of the moduli space, then you get a compact complex manifold divided by a finite group. So this vector bundle extends to this thing, and what happens is, ah, right, right. So in the genus two, with no marked points, in the limit, one limit, you get a torus with two marked points, you get the poles, with the sum of the residues is zero, that's your natural two dimensional space, that's sort of the limit of the stuff from the inside. When you break things apart you get bounding circles so the points are holomorphic.

I've been working on this picture for quite a few years, trying to understand one picture with another, and I found out how to avoid some of this stuff. There are a couple of papers in the Annals by [unintelligible]and Smiley and H. Mazer, now we want to go back to this picture here, related to the harmonic functions, the weights, so let's think about this now. I want to think about Riemann surfaces with two kinds of, since the sum is zero and I only have to worry about real parts, I've got positive and negative, so some are inputs, some are outputs. If one set is nonempty, the other is nonempty. I always need at least two points. Then we have a conformal structure, we know what a circle is, so there's a circle of directions around each of these points, it's almost metric, it has a natural metric defined up to scale, you have the rotation group of the conformal structure. Now let's think, this is supposed to be a morphism in a category, These are metric circles, the objects, and the outputs will be metric circles, one dimensional Riemannian manifolds, closed. And I can, a Riemann surface will define a morphism in this category, but I need to identify each circle with the circle of directions. So there's a circle ambiguity here. I want to preserve the metric structure.

Now I'll add the condition that the total length, the length of the circles, that's the energy of the string, so the total length on the input should be the same as the output.

[The identification of circles with directions is part of the data?]

Yes.

[Remove a point or remove a disk?]

I did not remove a disk.

[So you removed the point?]

Not necessarily. I don't want to remove a disk, create an infinite change of conformal structure, I'd never do that.

Now if I associate these things about this, the length gives me my weights that I didn't have before Then the total energy is the same, I get a unique, I'm trying to get an internal picture, I get this harmonic function well defined up to this additive constant, from these weights.

It turns out that this is a proper harmonic function. Now let's look at what's above it. Above every point in the real line is the surface, sperad out for us, it has these levels, and at a noncritical value it will be transversal and you'll get a compact manifold, since it's proper, so the manifold is foliated by circles outside of the critical values. There are only finitely many critical points, if we cut this off at a finite stage we have a finite cobordism between circles, and the critical points, there are no minimums and maximums, because it's harmonic, they're of intermediate type, normally they'd be Morse. The number of critical points is estimated by the topology except some can coalesce, it's the absolute value of the Euler characteristic, $2g - 2 + p + q$. You count multiple critical points with the appropriate multiplicity. You count foliation by the levels, you then have the orthogonal lines too, so you get a second foliation, the gradient flow lines. When you look out near infinity, the geometry of this is, near infinity, geometrically, I have these two orthogonal foliations, and in terms, this is length one, this is length one, a right circular cylinder.

[This only happens asymptotically?]

Exactly, you have cylinders coming in and going out, and the sum of the lengths is equal. What happens in the middle? At a critical level you have a graph, something, on the surface, and the vertices are the critical points of the function, and this is a harmonic function, the real part of a harmonic function, you know the levels look like that, if you have z^2 it's $x^2 - y^2$ and you have these two levels, $x = y, x = -y$. For z^3 you get this, so anyway, you get an even valence graph, what happen, when you look at the functions, it's like plus minus plus minus, if you look at this basic case, two circles are coming along, and two pieces come together and then they merge with this thing here and become part of a single level and then go off like this. There's a little cutting and reconnecting. Then these branches connect and become this circle down here, so it's this operation where two circles come together and you reconnect. In the more general one, three come together, you cut, and this piece goes with this piece and goes off like that, this piece goes with this piece and goes off like that,this piece goes with this piece and goes off like that. So total length is preserved, you cut and reconnect. Each morphism in this category has this internal picture.

[Why do you have higher valences if you assume the function is Morse?]

I don't assume it's Morse, then it would have valence four. What can happen? Two of these, these critical points might come together. This edge collapses. Then you get six. Two come together you get six, more you get eight.

[You might get singularities in this function that have nothing to do with the topology.]

Everything has to do with topology. This process going the other direction is what we used to call Morsification. Moduli space produces slightly generalized Morse functions.

The collapsing of the edge happens as I move through the modulus. I learned about squashing this technique in Princeton, a poor speaker, one person would ask him a question and knock him back like this, another person would ask a question and knock him back further.

If there's only one, if in moduli space there was only one critical level I could stop work on this five years ago. So, uh, actually, this is not completely trivial, because, see for example when you get an edge back to the same critical point, you can't collapse that edge.

So, uh,

[Do you have a local formula for h near the similarity?]

Yeah, $\lambda \log r$.

[What kind of graphs can you get?]

All graphs that are even valent, kind of like the conditions Veronique was writing down. These are cyclically ordered, even valence, the boundary divides two parts so that each boundary covers the graph exactly once. If you thicken this up locally you get an input-output thing locally here. They all occur. Now there's another picture that relates to, that's considering the vertical foliation, let's consider the other foliation, at the critical points, they're even transversal at the critical points. I won't draw the level again, the other one looks the same, as you get near here, the gradient lines, some go back like this, some go like that, two are coming in and two are going out. If you get near here you get half of these things coming down and landing on the things that are coming in. Then you actually see these generalized chord diagrams, you have circles, you add these chords, then what happens is as these circles move in you get these spiders.

[You're tracing the critical points back along the foliations?]

These spiders are waiting and as you come in they grab on and rip 'em and [whew] you get out again.

[Is this what Thurston used to call the interval exchange map?]

Exactly, just think of these, drop your topology, then there's a measure-preserving translation from here to here, they get cut into a lot of pieces and rearranged. So, uh, all right, so that's the, sort of internal picture of this category of Riemann surfaces. Now, I think we started five minutes late, I won't have time to introduce the propagator in the loop space.

We have a second gift from algebraic geometry, the nice picture of this bundle over the moduli space, so what can happen? I was describing the levels, but what does the Riemann surface look like? If two of these circles come together and become one, what you're doing is creating an actual pair of pants, two pieces of material, slit like this and sewn together. The apparent singularity, because I made the gradient have norm one arbitrarily close to where it's zero. Then thes cylinders can come together in more bunches. This is just a generalized cylinder running across. So on this side, I need a little bit more room, So what you have for example is a cylinder, and this has stuff in it, this stuff is coming along and a relatively small cylinder goes across, this is slightly bigger on bottom. In the limit of this, you get these two surfaces and these points appear and positive and negative weights appear, I think of this as exchanging a photon, a piece breaks off and goes to the other one.

In the real vector bundle in the limit, you get these weights appearing at these points. You could actually have the thing reversed, this would be positive and this negative, and the weights could be zero, it wouldn't exactly look like that. The stuff on these two sides is united for a moment and then it breaks apart. You can extend this picture of what's happening beyond the moduli space to the completion.

I never proved this but always felt it was true, that these operate on the moduli space, but they extend beyond. I don't know whether you need more structure, Virasora algebra structure or something, but this should extend beyond.

I guess I'll do all of the string topology next time.

[What do you mean "extend the string topology beyond the moduli space?" Where beyond?]

That question is sort of out of order, I haven't explained what I meant by string topology. The operations may be organized by a compactified version of moduli space, beyond the open part, beyond open moduli space. Now I'll put chains, loops in there, the data has interesting operations to infinity. The limit is the point circle, that makes sense, if I can understand the limiting pictures that way,...

[There is also a way using trivalent graphs, what is the relation?]

What's the question, the math question? These graphs here don't determine everything. It's a fat graph with additional data. One question which you can ask about both fat graphs, this is not known, this approach goes back to Hilbert, there's a paper by Wolpert-Giddings with a cell decomposition of moduli space, Green's function, let's stay in math as much as we can, maybe Green was a physicist, haha, and there's the Thurston picture, which actually goes back, Riemann surface people knew this picture, Thurston made it very clear. This allows you to understand the relationship between the moduli space and its compactification. A math question is, given a fat graph, find the thick thin decomposition, or the other way, I understand some, but not all of it.

[This gives you a cell decomposition of moduli space. Does it extend to the compactification?]

There is no known cell decomposition of open moduli space that extends cell to cell to the compactification, although Søren was telling me about something at the AIM.

By the way, there's a lot of confusion, there are two fat graph approaches, there are two distinct proofs, the transcendental holomorphic and the hyperbolic geometric semi-algebraic decomposition, they give different decompositions labelled by the same things. There are two maps, take a point in moduli space, use the Penner construction to get a point in the fat graph cell complex and then by Streibel, the fat graph can be interpreted as having a canonical formal structure, and then you get something that is not the identity. A natural question is to show that it is a bounded distance from the identity in the quasiconformal metric. That's doable but it hasn't been done. The easy map is in opposite directions. The inverse maps are hard. It's very very different. What's good about the two approaches, you can go in some direction. That's why I was tough on your question, there are some interesting math questions but they're quite hard.

3 Fukaya

[There is still lunch. The bus comes at around 2:30]

Thank you very much, I want to discuss some technical point to work out mathematical details of the construction of TFT. I'm sorry that it's not a good subject to talk at the end of the day. I want to talk about transversality.

I want to explain the problem. What I mean by TFT, I don't mean Atiyah, this is much more general. Start from a manifold M. You fix some data, some system of moduli spaces $M_\rho(M)$. You have a map from this space to a product of the space you started with, then you have a correspondence to get some algebra $C(M)$. This is some kind of process. By topological, you need various choices, like maybe a metric or an almost complex structure or Morse functions, and then to do the second step you need perturbations for transversality and maybe some other things. Topological means $C(M)$ is well defined up to homotopy equivalence. This means various things in various contexts.

My problem is something here, I want to address a problem, a general problem about transversality. How do you make the correspondence rigorous? If you do it naively you get something highly singular. I want to show you two examples.

1. $L^n \subset M^{2n}$ is a symplectic manifold, Kahler, with J an olmost complex structur and ω

with $d\omega = 2$ and $\omega^n \neq 0$. Then $\beta \in H_i(M, L)$ and $M_\beta = \{\phi : (D^2, \delta) \to (M, L) | [\phi] = \beta, \text{holomorphic}\}/G$. Here $G = \{U : D^2 \to D^2 |, \text{biholomorphic} U(1) = 1\}$. So $M_\beta \to \mathscr{L}L \subset \{\ell : S^1 \to M\}$ and $\phi \mapsto \phi_\delta$.

Theorem 2

$$\delta M_\beta + \sum_k \sum_{\sum \beta_i = \beta} \{M_{\beta_i}\}_k = 0$$

This is somehow the deformation of some string topology. To make this rigorous you need transversality, the virtual fundamental chain. This is one thing, there is another thing, this is written in a preprint, that's somehow more elementary.

[Can you explain, the right hand one is a chain in the loop space? The left is as well?]

Yes, which chain complex you take is another headache. The very similar equation is the following thing

2. L a C^∞ manifold compact finite dimensional. A Stasheff cell M^0_{n+1} is $\{(z_0, \ldots z_n) \in (S^1)^{n+1} | \text{respects cyclic order}\}/PSL(2, \mathbb{R})$. And δM^0_{n+1} is $\sum_{1 \le i \le j \le n} M_{j-i+1} \times M_{n-i+j+1}$. Then $M_{n+1}(L) := L \times M_{n+1}$. Then I want to use the correspondence L^n pulls back to $M_{n+1}(L)$ and pushes down to L.

Theorem 3 *M_n define an A_∞ algebra*

[This is really what makes a mathematician tick.]

I want to show some difficulty of the transversality business. You have B the set of all H_2 and if you have a β you can say that the set of $\alpha \in B$ less than β are finite. You use B as a partially ordered set for inductions. So you have M_β maps to $M_\beta(L)$ and also to $M_{n+1}(L)$. Then ∂M_β is a fiber product of M_α for $\alpha < \beta$. This is not just a direct product, it's a fiber product.

If you take $\delta M_{n+1}(L) = \cup_{i,j} M_{j-i+1}(L) \times_L M_{n-i+j+1}(L)$.

So let me explain what is the problem. In general the moduli space can be quite singular, let me cheat, this is contained in a smooth manifold $M_\beta \subset U(\beta)$ with sections S_β to $E(\beta)$ and the this is $S_\beta^{-1}(0)$. Locally you have this picture with some glue in an appropriate sense. You cannot expect that $S_\beta \pitchfork 0$. What's nice is the following, suppose you have $\delta M_\beta \ni M_{\alpha_1} \times_L M_{\alpha_2} \subset U_{\alpha_1} \times_L U_{\alpha_2}$ and the fiber product here is transversal.

Then this one $\delta M_\beta \subset U_\beta$ and we can identify

$$\begin{tikzcd} U_\beta \arrow[d, shift right, "S_\beta"'] & U_{\alpha_1} \times_L U_{\alpha_2} \arrow[l] \arrow[d, shift right=2, "S_{\alpha_1}"'] \arrow[d, shift left=2, "S_{\alpha_2}"] \\ E_\beta \arrow[u, shift right] & E_{\alpha_1} \times E_{\alpha_2} \arrow[l] \arrow[u] \end{tikzcd}$$

The problem is can one find S^ϵ_β such that S^ϵ_β is a perturbation of S_β and $S^\epsilon_\beta|_{U_{\alpha_1}\times_L U_{\alpha_2}} = S^\epsilon_{\alpha_1} \oplus S^\epsilon_{\alpha_2}$ and $S^\epsilon_\beta \pitchfork 0$. We cannot do this in general.

[Some discussion of why this cannot be done in general.]

Of course you can do this if you perturb everything without compatibility. Let me say that in some situations we can apply different perturbations to different parts of moduli space. [unintelligible]mathematical detail of the Gromov-Witten invariants.

The case [unintelligible]is when you have something only well-defined up to chain homotopy. If all the structure constants are well-defined then this problem disappears, what is well defined is the homotopy type of the full structure.

This looks like a highly technical problem but it is quite important.

There are two arguments, I want to explain the first briefly and go into detail on the second one.

One way is the following things. We consider a continuous family of perturbations S^w_β for $w \in W$ some measure. Then we consider $\int (S^w_\beta)^{-1}(0) d[unintelligible]w$.

So we have $(S^w_\alpha)^{-1}(0) \times_L (S^{w'}_\alpha)^{-1}(0)$. The diagonal is measure zero, so we can do this. In this way we can keep any symmetry we want. The bad point is that we can only work over the real numbers. If you want integers or torsion, you are forbidden to take this happy way. For some applications, we want to work over $\mathbb{Z}$ coefficients or $\mathbb{Z}_2$ coefficients.

So I want to explain the other way to work, over integers but breaking many symmetries. The second case, $M_{n+1}(L) = M_{n+1} \times L$, and this maps to L^n and L. The problem is to perturb $M_{n+1}(L)$ over the integers. I want to remind you that the moduli space is not so bad, but the map to L^n is not transversal. You pull back and push back, this is hard to do if it is not transversal.

The other somehow delicate thing, I want to say something technical so I need to make it rigorous. I need to say which chains to choose. This is our choice. Soy $S(L,\mathbb{Z})$ is the smooth singular chain complex. It's (P,f) where P is a manifold with or without corners and $f : P \to L$ is C^∞. Technically this is not so good. I want to map this to $(\wedge(L))^\dagger$, so this goes to $(u \mapsto \int_P f^*(u))$. So the image of this map is $\bar{S}(L,\mathbb{Z})$.

probably this theorem is harder to state than prove, but I want to say it.

Theorem 4 *For each positive real N there exists a chain complex $C^1_*(L) \oplus \cdots \oplus C^n_*(L) \subset \bar{S}(L,\mathbb{Z})$. So $C^{(g)}(L) = \oplus_1^g C^i(L)$, and there exist perturbations $M_{n+1}(L)^{(g_1,\ldots,g_n)}$ with $g_i \in \{1,\ldots,N\}$. This is of $M_{n+1}(L)$. This will be an embedding but it doesn't matter.*

Conditions:

1. *$C^{(S)}(L)$ is a subcomplex with homology the same as $H(\bar{S}(L))$.*
2. *$[P_i, f_i] \in C^{g_i}(L)$ and $(P_1 \times \cdots \times P_n)_{(f_1,\ldots,f_n)} \times_{(ev_1,\ldots,ev_n)} M_{n+1}(L)^{g_1,\ldots,g_n}$ is transversal.*

I will write this $M_n(P_1, \ldots, P_n)$.

3. $(M_n(P_1, \ldots, P_n), ev_0) \in C^{(g)}(L)$, $g = \max\ g_i + n$.
4. $\cup C^{(s)}(L) = C(L)$

Then the claim is that M_n defines an A_n-algebra on $C(L)$.

You can guess how to prove this, by induction.

[Discussion of this.] The difficult part is to work out the combinatorics to see that your induction works. You want the boundary of the fiber product to agree with the A_∞ relations. You have to check the way you order your induction to see how it works. So you can extend it to the interior. In this way you make everything transversal.

Since I have two minutes left, let me talk about A_n to A_∞. We kill everything in a neighborhood of the diameter. But for the diameter, you compose n times you get $n\epsilon$ the neighborhood of the diameter. When n goes to infinity you lose control. So this is the reason we fix n first and construct an A_n structure. So the thing is the following, for each N we get $(C^{[N]}(L), m_*)$ an A_N structure, and so you take (C, m) an A_N algebra and for $N' > N$ you get (C', m') an $A_{N'}$ structure, then these are A_N homotopy equivalent. Now there is some lemma

Lemma 1 *If C, C' are N, N' structures and f is an A_N-homotopy equivalence, $N < N'$, then the A_N-structure on C extends to an $A_{N'}$ structure and f extends to $A_{N'}$ homotopy equivalence.*

Since the structure C' extends you get this f. For each N you get an N structure, and then you extend to N'.

This was all I wanted to say.

[So you and Ralph wrote about the Morse theory story, where you need two carefully chosen Morse functions. Is there any sense in which what you're doing is analogous to that problem of commutativity?]

What we can prove is that this A_∞ category is equivalent to [unintelligible], but the transversality issue is very serious, I think it's too difficult.

[Is this an analogous problem?]

I don't think so. Maybe you can, I think it's difficult. Ralph?

[You might be able to get past it, you are trying to make an umkehr map, if they live over smooth manifolds.]

[He's trying to do something finer.]

[I agree.]

Probably you can try to do it on the chain level.

[A very naive question, can you get an obstruction bundle, why doesn't that work?]

This is the same answer.

[He wants a manifold with boundary with its boundary being related to other structures.]