Stringy Topology Notes
January 17, 2006

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We need to know if people are interested in the excursion today. We have to have the bus company, a quick poll, are many people interested in the excursion.

All the travel reimbursements are by MSRI. For trainees only, that were offered money, we have your money here. Everybody else you have to get it from MSRI.

1 Teichner

Good morning, I'll try my best to warm you up. I can walk around and stretch, I have a slight advantage here.

I'm going to first recall some things I said yesterday. I'll start with a reminder of the EFT version of K-theory and then do an analogy of a CFT version for "elliptic cohomology." I'll try to draw analogues of everything.

1.1 Reminder of the EFT version of K-theory

Let me remind you of the space $\mathrm{EFT}_m = \{D \text{ on } \mathscr{H}\}$, where $\mathscr{H}$ is a graded Cl_n-algebra. These were odd self-adjoint Cl_n-linear and e^{-D^2} is compact. Up to homotopy type you can put finite rank or anything in between.

Because of the condition here this is a very easy space. The eigenvalues are real, the eigenvalues are discrete and the eigenspaces are finite dimensional. Because of linearity the eigenspaces are Clifford modules. The eigenspace at $-\lambda$ by oddness is the grading involution of the eigenspace at λ, so the 0-eigenspace is a $\mathbb{Z}_2$-graded Clifford module.

There's a nice map into finite dimensional graded Clifford modules by $D \to \ker\ D$. If I mod out by those that extend to Clifford $n+1$-modules (graded) then I get an isomorphism.

[What is the supersymmetry here?]

It is just that D is odd. This is the easy version. So this map, if you have a configuration, you can push anything that's not in the kernel to infinity. Then if there's a configuration, I should say, the topology comes from, if two points move together, the eigenspaces add, or eigenspaces can move to infinity and vanish. They're all orthogonal and their sum is not $\mathscr{H}$. If two parts come together at 0, they have to come in on both sides simultaneously. If you allow such homotopies you get the quotient by the $n+1$ modules. You probably remember that this is the Atiyah-Bott-Shapiro version of $KO_n(pt)$.

[What is ∞?]

D is very unbounded, if $D = \infty$ then e^{-D^2} is compact in the usual sense. Instead of $\pi_0(EFT_n)$ you take homotopy classes of [unintelligible].

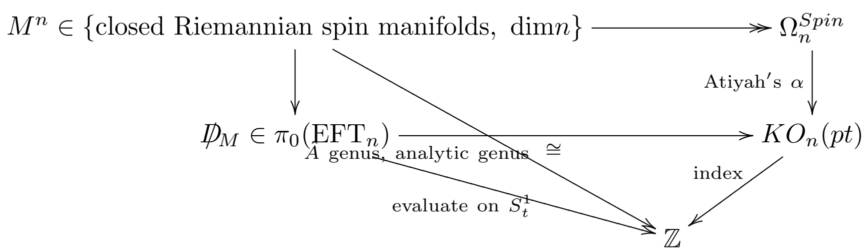
What we did, we made this simple space complicated by introducing EFTs, we did this because people tried to generalize the simple version and failed.

The other definition I wanted to remind you of is that this space EFT_n is also the space of all 1-dimensional supersymmetric Euclidean Field theories of defree n. If you have a field theory E then $E(pt) = \mathscr{H}$. Then degree n means you get graded Cl_n-modules. You can also study $E([0,t])$. You get e^{-tD^2}. You get this if you're not supersymmetric, you will also get some D if you are supersymmetric.

One nice thing is, you can take a field theory and evaluate it also on a circle of length t, S^1_t. So $E(S^1_t) \in \mathbb{R}$. The beauty is that if you evaluate an EFT on S^1_t it will not depend on t and it's going to be an integer. This real number is just the supertrace $str(E([0,t]))$. This follows from the axioms of a field theory. The very first argument in a physics book on this will say that that this is independent of t and an integer. It's the index from KO_n $n \equiv 0 \pmod 4$ or the superdimension.

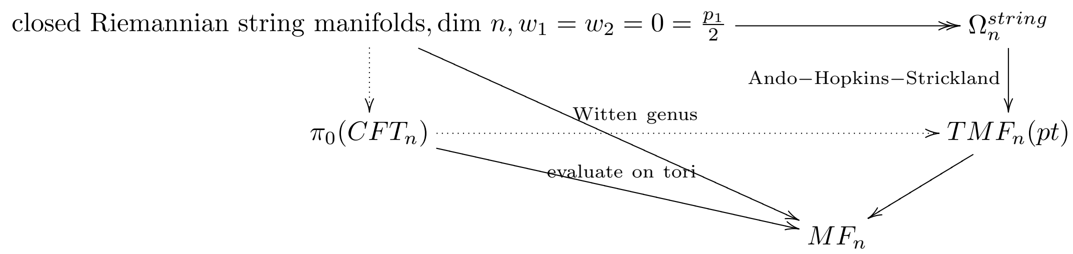
[Strictly speaking this only works for $n = 0$. Also the circle needs a spin structure.]

Now let me enlarge my diagram a little bit.



1.2

So now say we have



This theory of topological modular forms maps to every elliptic cohomology theory uniquely. It's not elliptic and is called this because it maps to modular forms.

The Witten genus is the index of the Dirac operator on the loop space, $\text{index}^{S^1}(\not{D}_{LM})$. The way he got to this is, he, and this is a pleasure to write this down again, he used the Atiyah-Bott fixed point theorem to calculate the index, the fixed point is the manifold. That formula makes perfect sense and gives you the genus. The refinement from Ω_n^{string} is a lifting of the Witten genus to something much more interesting. One of the points of trying to make the space $\pi_0(CFT_n)$ is making a lot of new index theorems, we can't do this yet because the arrows are dotted. Just as you can think of the index as information about the EFT, you can get it from information about the CFT. So Witten had this whole quantum field theory in mind. This is just one part of that. It's the supertrace associated to the annulus of a certain length. There are other operators corresponding to other surfaces. This is how you think of the Dirac operator, restrict to this surface and take the infinitessimal generators. The infinitessimal generator for $E(\text{annulus}) =$"$\text{index}^{S^1}(\not{D}_{LM})$"$\subset \sigma$-model of M.

Before the amazing thing was that the evaluation was an integer, here it is an integral modular form.

I think I made the analogy pretty well, I hate to erase this.

Now $MF_* :=$ integral (weak) modular forms. A modular form of weight k is a holomorphic section of $Det^{\otimes k} \to \mathscr{M}_{tori} \cong S^2$, which contains a cusp $\tau = i\infty$, we write $q = e^{2\pi i \tau}$.

Integral means you have integer coefficients in your q-expansion and weak means it's only meromorphic at the cusp.

So you get $MF_* \cong \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]$ with $c_4^3 = c_6^2 = 1728\Delta$.

I should say the degree n is defined as $2k$, twice the weight. This is because $Det^{\otimes k} \overset{\text{spin str.}}{=} Pf^{\otimes 2k}$. This is again only for dimensions divisible by four.

The $TMF \to MF$ is a rational isomorphism.

This was the easy part. Instead of working over arbitrary rings, just use supersymmetry and you get the solid arrow, that's the punchline.

1.3

Definition 1 CFT_n *is the space of all 2-dimensional extended supersymmetric CFTs of degree n.*

So a regular CFT is a functor from the 2-dimensional bordism category of conformal structures to $Vect$, so that is everything but three words.

Supersymmetry is what gives you holomorphic and the map by evaluating on tori. The definition will be tomorrow.

Now this business about extended. First the definition and then the why. We really have a functor from a cobordism 2-category $CB^2 \to Vect^2$. You have 0-morphisms, 1-morphisms, and 2-morphisms, I'm not going to take you through the definition, but the objects are 0-dimensional conformal manifolds, the morphisms 1-dimensional conformal manifolds, and 2-morphisms 2-dimensional conformal manifolds. These have corners. If you just do the gluings you can do, you'll rediscover the definition of a 2-category.

In $Vect$ we also want to extend. We'll take von Neumann algebras as objects, bimodules as morphisms and intertwiners as 2-morphisms. I don't want to say too much about this aspect but the reason we want this is

Conjecture 1 $\Omega\mathrm{CFT}_n \cong \mathrm{CFT}_{n+1}$

So we think we can define an Ω-spectrum, but only if we use the extended version of a CFT. What I really want to concentrate on is the degree n. This will build in the correct weight of a modular form and be in analogy to the Clifford module structure in the K-theory.

[You can make a space?]

Just as we fixed a Hilbert space we'll fix a von Neumann algebra and a bimodule.

E has degree n if

- $E(X^1)$ is a module, $\mathbb{Z}_2$-graded, over $C(X)^{\otimes n}$, where $C(x) := \mathbb{C}l(\text{spinors of } X)$.

  [What's the inner product on the spinors? Oh, they happen to have one on a 1-manifold.]

  So the analogy is that $C(pt) = Cl_1$ and $C(pt)^{\otimes n} = Cl_n$. This is cheating a little, it's really a superpoint, a good way to excuse yourself.

  Now the other thing about the degree is, well, if Σ^2 is a conformal surface, there is a canonical graded irreducible module over the Clifford algebra over $C(\delta\Sigma)$ namely $\mathrm{Fock}(\Sigma)$. Even for Σ closed you get $\mathbb{C}$ and this is $Pf(\Sigma)$ which is still interesting. You have to take restrictions of the [unintelligible]of spinors to the boundary, that's a Lagrangian, [unintelligible].

I should have explained things better, I'm rushing. The degree, it's a module of this Clifford algebra tensor n.

- Really only $E(\Sigma, \psi) \in E(\delta\Sigma)$ is defined for $\psi \in \mathrm{Fock}(\Sigma)^{\otimes n}$ and the association $(\Sigma, \psi) \mapsto E(\Sigma, \psi)$ is $C(X)^{\otimes n}$-linear in ψ.

  Maybe I should point out that the Fock space has a canonical element, the vacuum element. You can reduce to that element but somehow that's not quite right.

  Note that on the torus we really get a degree n modular form. You get a number, and an element of something, you get a section of the dual power of the Pfaffian line.

Exercise 1 *This is the punchline, I will explain it to anyone who asks. For K-theory this is equivalent to*$E([0,t]) \in \mathscr{K}_{Cl_n}(\mathscr{H}) \subset \mathscr{K}(\mathscr{H})$.

Sorry for going over.

[Yesterday an important role was played by EFTs being a monoidal functor. On the left, well, these are symmetric monoidal bicategories, so I should read [unintelligible]'s 200-page definition of this.]

Just like in K-theory it came down to understanding intervals. Now it comes down to triangles, but they have an interesting moduli space. This whole machinery will reduce in the end, you only have one bicategory to understand, that has the right structures.