Stringy Topology Notes
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1 Teichner

I do not take notes for lectures with transparencies.

There is an excursion on Wednesday to Patzcuoro. Get your tickets and lunch tickets in advance.

The definition of a lunch is different in Mexico, it is the main meal of the day.

2 Galatius, Graphs and the Homology of $Aut(F_n)$

Uh, thank you, I don't know if there's anything string about this, but $Aut(F_n)$ is the automorphism group of $\langle x_1, \ldots, x_n\rangle$.

[(As he attempts to untangle wires) I'm sorry, I encountered a nontrivial element of a fundamental group.]

Many people have used graphs to study groups like this. I'll study $Out(F_n)$ by analogy to mapping class groups.

$\Gamma_g = \pi_0(\mathrm{Diff}(\Sigma_g))$,
Teichmueller space
Moduli space of curves
$\Gamma_{g,n}$ which is π_0Diff of surfaces with n boundary components, relative to the boundary
Cobordism category (Segal's category)
So I'm going to study, for the mapping class group it's convenient to let g go to infinity, so you have stability b
Mumford conjecture $H^*(\Gamma_{g,n}, \mathbb{Q}) = \mathbb{Q}(x_1, x_2, \ldots)$ π for $* \ll \infty$
Madson Weiss Theorem proving this, $\mathbb{Z} \times B\Gamma^+_{\infty,1} \cong \Omega^\infty \mathbb{CP}^\infty_{-1}$
A slightly stronger

Corollary 1 *$H_*(\Sigma_n) \to H_*(Aut(F_n))$ is an isomorphism (integrally) in the stable range.*

First I should say why I can't just copy the Madsen-Weiss theorem. The main point of what goes wrong is that the map uses the Pontryagin-Thom collapse map, there's no analogue on the right-hand side. The proof (that Weiss gave?) uses the h-principles. Both of these rely strongly on the object being smooth, so we have to do something else.

[Is there any anologue to replacing homotopy equivalences on the right with geometric equivalences?]

The quotient of the moduli space of graphs by the contractible outer space. You have fibers that are graphs but as you move around you will collapse edges. This will be part of the proof, to work with what you could call graph bundles.

That was the introduction.

2.1 The space of graphs in $\mathbb{R}^n$

A graph in $\mathbb{R}^n$ is subset G of $\mathbb{R}^n$ such that for every $p \in \mathbb{R}^n$ there is a neighborhood $U_p \subset \mathbb{R}^n$ such that $U_p \cap G$ is either

- empty
- the image of a smooth closed embedding $\gamma : \mathbb{R} \hookrightarrow U_p$
- the image of a smooth closed embedding $\vee^k[0, \infty) \hookrightarrow U_p$ for $k \geq 3$.

(the second and third correspond to edge and vertex points.)

This is not necessarily compact or finite, but locally finite. Let $\Phi(\mathbb{R}^n)$ be the set of graphs in $\mathbb{R}^n$ Then $\Phi(\mathbb{R}^n)$ has a natural topology. I'm not going to write down the open sets, but it allows continuous edge collapses, so that if I have a sequence of graphs where an edge length approaches zero, in the limit the edge will collapse. There is also a continuous push to "infinity," cutting an edge by pushing its center to infinity.

[Can you collapse an infinite sequence of edges as you go off to infinity at the same time?]

What? I specify a neighborhood, I draw a compact manifold. A codimension zero compact manifold whose boundary intersects the edges transversally, [unintelligible]. It's the topology of what they look like.

That's not a definition but a description. So, comments.

1. $\{G \in \Phi(\mathbb{R}^n) | G \subset \mathring{I}^n\} \subset \Phi(\mathbb{R}^n)$. Then the homotopy type of G is locally constant. If I bound with the unit cube I can't push anything to infinity any more. This isn't a

closed subspace. It might be. It's definitely not open because in a neighborhood of something you have elements arbitrarily close to infinity.

So components of $\overset{g}{V}\ s'$ gives a model of $B\ Out(F_g)$ as $n \to \infty$. I allow things to move around and collapse edges. This is just the statement that both classify families of graphs. The appropriate component is Whitney embedding, the $B\ Out$ statement is what [unintelligible]and Vogtmann proved. It's analogous to the model of B Diff where this is $Emb(M, \mathbb{R}^\infty)/\mathrm{Diff}(M)$.

2. Similar models of BA_g^{n+m}. You take graphs with incoming and outgoing leaves on a boundary of a neighborhood.

3. The spaces $\Phi(\mathbb{R}^n)$ are analogues of Thom spaces.

   If I define, to explain the analogy, define $\Psi(\mathbb{R}^n)$ as the space of all d-manifolds then it's not to hard to see $\Psi(\mathbb{R}^n) \xleftarrow{\simeq} \mathrm{Th}(U^\perp \searrow \mathrm{Gr}_d(\mathbb{R}^n)$ by $L + v \leftarrow (L, v), \emptyset \leftarrow \infty$.

4. The $\Phi(\mathbb{R}^n)$ form a spectrum which factors as:

$$\begin{tikzcd}
\mathbb{R} \times \Phi(\mathbb{R}^n) \arrow[rr] & & \Phi(\mathbb{R}^{n+1}) \\
 & S^1 \wedge \Phi(\mathbb{R}^n) &
\end{tikzcd}$$

   by $(t, G) \mapsto \{t\} \times G$.

5. There is an analogue of the Pontryagin-Thom collapse. Remember the spaces that gave me, that were models for $B\ Out(G)$, $\{G \in \Phi(\mathbb{R}) | G \subset \mathring{I^n}\}$. This maps to $\Gamma((I^n, \delta I^n), \Phi^{fib}(T\mathbb{R}^n))$.

   So note that Φ, it has more structure than being a space, it's a sheaf on $\mathbb{R}^n$. It's not just a sheaf, it's funtorial under embeddings. So a sheaf by definition is functorial under inclusions, but here I can embed as well.

   In particular, $\mathrm{Diff}(U)$ acts on $\Phi(U)$. All of this is compatible with the topology. So the diffeomorphism group acts continuously. The tangent bundle of $\mathbb{R}^n$ is a [unintelligible]with structure group $\mathrm{Diff}(\mathbb{R}^n)$.

   [I don't know what that symbol means, can you make the math statement? I'm willing to believe any theorem, I just want the statement.]

   $\Phi^{fib}(T\mathbb{R}^n)$ is a fiber bunder over $\mathbb{R}^n$ with fiber $\Phi(T_p\mathbb{R}^n)$.

   So the map, the arrow, I haven't defined it yet, is induced by an exponential map on $\mathbb{R}^n$, so how does it work? Given G in the first space and p in the unit cube, I have $T_p\mathbb{R}^n \hookrightarrow \mathbb{R}^n$, so I use functoriality $(exp_p)^*G \in \Phi(T_p\mathbb{R}^n)$.

   This would not be a continuous map if you couldn't move things out to infinity and so on.

   The object is cross sections of the sheafs of local versions of the object. You can think of this map as an h-principle style map. The range space is, of course, the bundle is trivial, so this is $\Omega^n\Phi(\mathbb{R}^n)$. So as I let n go to infinity, I get $B\ Out(F_n) \to \Omega^\infty \lneqq$.

I have two minutes left. You get similar maps from the slightly more general spaces that form the morphism spaces of the category. The two remaining steps to prove the theorem are, very roughly,

- use Gromov's h-principle for flexible sheaves, and
- Prove $\lneqq \cong S^0$

The first steb is to say that Φ is microflexible, then $\{G \in \Phi(\mathbb{R}^n) | G \subset \mathbb{R}\times \mathring{I}^n\}$ maps to $\Gamma((\mathbb{R}\times I^{n-1}, \mathbb{R}\times \delta I^{n-1}), \quad) \cong \Omega^{n-1}\Phi(\mathbb{R}^n)$.

[Some explanation of this part, and explication of the difficulties involved.]

To get $\mathbb{Z}\times B\ Aut_{\infty}^{+} \cong \Omega^{\infty}\lneqq$ from this is as for manifolds.

That's it.

[So, this last couple steps, the original theorem is about a homology equivalence, does this mean this space you've written down is a model for group completion?]

[unintelligible], Yes

[What is the definition of $\lneqq$?]

This is the spectrum of $\lneqq(\mathbb{R}^n)$.

[How much of this can you do for higher dimensional manifolds with nodal singularities?]

Everything, probably, as long as it's characterized by a local property, but you won't have homotopy type of S^0.

3 Tillman, cobordism categories and their classifying spaces

Thanks to the organizers for inviting me.

[Was it explained to the students, the classifying space of a category?]

Yes. These became important when Atiyah-Segal realized that quantum field theories could be realized as functors from these to Vect. One way to get a grip on such categories is by associating a space. So $\mathscr{C}$ is a category, $B\mathscr{C}$ is a space. How much of $\mathscr{C}$ do you remember? When the good completion applies, the morphisms are reflected in the space. In other cases, you hope to get something different. The main theorem today is to realize a d-dimensional cobordism category as a Thom spectrum.

Start with $\mathscr{C}_d$, the cobordism categord of $d-1$-dimensional smooth closed compact manifolds, and I will think of them as embedded in $\mathbb{R}^\infty$, and their cobordisms (of dimension d, imbedded in $\mathbb{R}^\infty \times [a_0, a_1]$.) Here is the picture. By the Whitney embedding theory, the objects are disjoint unions of diffeomorphism classes of $d-1$ dimensional manifolds. So this is a model for the classifying space $B\mathrm{Diff}(M)$. And the morphisms are disjoint unions of diffeomorphism classes of W^d thought of as bordisms. So I have ingoing and outgoing boundary components. So I want to distinguish between ingoing and outgoing boundary components

[Speaker backs loudly into the television]

So there's an oriented version $\mathscr{C}_d^+$ and other versions. Consider $\mathscr{C}_{d,b}$ where every connected component of W has at least one outgoing boundary. Of course there is also $\mathscr{C}_{d,d}^+$.

Theorem 1 *(Galatius, Madsen, T., Weiss)*
There are homotopy equivalences $\alpha : B\mathscr{C}_d \xrightarrow{\sim} \Omega^\infty\Sigma\mathbb{G}_{-d}$, $\alpha : B\mathscr{C}_d^+ \xrightarrow{\sim} \Omega^\infty\Sigma\mathbb{G}_{-d}^+$.

Just by writing this, these had better be infinite loop spaces. This is induced by the symmetric monoidal structure structure on the category. We have to use some machinery, and you also need the connectod components to be a group. Look at $\pi_0 B\mathscr{C}_d$. As soon as, the connected components are the points in the object space. If they are cobordant there is a path from one to the other in the classifying space. So this $\pi_0 B\mathscr{C}_d$ is Ω_{d-1} of the cobordism group in the sense of Thom.

The definition of $\mathbb{G}_{-d}$, it's a spctrum, it's a Thom spectrum, it's "$\Sigma_{-d} MO_d$."

Well, $G(d,n)$ is the Grassmanian of d-dimensional flanes V in $\mathbb{R}^{d+n}$ and $U_{d,n}^\perp$ is $\{(V,v) | V \in G(d,n), v \in V^\perp\}$. Now by adding $\mathbb{R}$ to the trivial bundle I get $\Sigma\mathrm{Th}(U_{d,n}^\perp) = \mathrm{Th}(U_{d,n}^\perp \oplus \mathbb{R}) \to Th(U_{d,n+1}^\perp)$ so this is a spectrum $\mathbb{G}_{-d}$. Then

$$\Omega^\infty \mathbb{G}_{-d} = \lim_{n\to\infty} \Omega^{d+n}\mathrm{Th}(U_{d,n}^\perp).$$

An example, $\Omega^\infty\mathbb{G}_{-0} = \Omega^\infty S^\infty$,
$\Omega^\infty\mathbb{G}_{-1} = \Omega^\infty\mathbb{RP}_{-1}^\infty$
$\Omega^\infty\mathbb{G}_{-0}^+ = \Omega\Omega^\infty S^\infty$,
$\Omega^\infty\mathbb{G}_{-1}^+ = \Omega^\infty\mathbb{CP}_{-1}^\infty$.

So we have a fibration $\Omega^\infty\mathbb{G}_{-d} \to \Omega^\infty S^\infty((BO_d)_+) \to \Omega^\infty\mathbb{G}_{-(d+1)}$.

I'm filling in the gaps that Sørin left. I need to define α.

Definition 1 *Remember we had the cobordism W. Choose* N_W *a neighborhood of it. This sits now in* $[a_0, a_1] \times \mathbb{R}^{d+n-1}$. *From here we get a map* $[a_0, a_1] \times S^{d+n-1} \to \mathrm{Th}(N_W)$ *(the Thom collapse map, identifying fibers at infinity, and sayning that outside the neighborhood, you map to infinity, then push forward tho the Thom space of* $U_{d,n}^I$, *something about* T_xW, V.

What I really get is a functor $\mathscr{C}_d$ *into the path category of* $\Omega^\infty\Sigma\mathbb{G}_{-d}$. *This gives a map* $\alpha : B\mathscr{C}_d \to \Omega^\infty\Sigma\mathbb{G}_{-d}$.

So we know of course examples of this already. When $d = 0$ this is really quite classical. The objects are empty and the cobordisms are configurations of k points in M. So this is homotopy equivalent to $\coprod B\Sigma_k^1$. So what does the Thom collapse map do here, we want $\Omega^\infty S^\infty = \Omega^\infty \mathbb{G}_{-0}$, which comes from $\Omega B(\coprod B\Sigma_k) \to \mathbb{Z} \times B\Sigma_\infty^+$ via α. This is Segal's point of view of Barratt, Priddy, Quillen.

In dimension two this is Madsen-Weiss, $\mathbb{Z} \times B\Gamma_\infty^+ \cong \Omega^i nfty \mathbb{CP}_{-1}^\infty = \Omega^\infty \mathbb{G}_{-2}^+$. and T : $\mathbb{Z} \times B\Gamma_\infty^+ \cong \Omega B\mathscr{C}_{2,b}^+ \to \Omega B\mathscr{C}_2^+ \xleftarrow{\simeq} \Omega B\mathscr{C}_{2,b}^+$.

[You can make both boundaries nonempty?]

This might not be true in higher dimensions. Basically we expect this last map to be true in higher dimensions. I think the double boundary part will have a problem, but I would expect that it might be true. The classifying space is quite rough.

[This second theorem is easier than Madsen Weiss?]

Yes.

Before I go into the proof of the main theorem, let me mention another category, sometimes called an HQFT, somewhere between a QFT and TQFT, where you have a mapping class group action. Here's a $d = 4$ example, Jeff Giatsecusa, called $2\mathscr{C}$. The objects are $\mathbb{N}$, thought of as n copies of S^3, morphisms are disjoint unions of simply connected oriented cobordisms between n and 1, and the 2-morphisms from $W, \tilde{W}$ are $\pi_0 \mathrm{Diff}(W, \tilde{W}; \delta)$.

Theorem 2 $\Omega B 2\mathscr{C} \cong \mathbb{Z}^2 \times BO_{\infty,\infty}(\mathbb{Z})^+$.

He had to replace Harer stability with $\Gamma(W^+) = \Gamma^4(W \backslash \coprod D^4)$ as long as $W = \tilde{W} \# \mathbb{CP}^2$ or $W = \tilde{W} \# \overline{\mathbb{CP}}^2$.
Then $\Gamma_\infty(W)_{\#(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2)} = O_{\infty,\infty}(\mathbb{Z})$.

The sketch of the proof. Let me briefly review $[X, \Omega^\infty \Sigma \mathbb{G}_{-d}]$. If I can show this is $[X, B\mathscr{C}_d]$ then I have proved my theorem.

If I have g in this, it is $g : X \times \mathbb{R}^{d-1+n} \to \mathrm{Th}(U_{d,n}^\perp)$. Then the preimage of $G(d, n)$ is a manifold M. Then $TM \oplus \mathbb{R}^{d+n}$ can be written $TM \oplus g^* U_{d,n}^\perp \oplus g^*(U_{d,n}) = TM \oplus \nu M \oplus g^*(U_{d,n}) = TX|_M \oplus \mathbb{R}^{d-1+n} \oplus g^*(U_{d,n})$. Then obstruction theory says I can find an isomorphism $TM \oplus \mathbb{R} \overset{\sigma}{\cong} TX|_M \oplus g^*(U_{d,n})$.

[X is a manifold?]

Yes, with empty boundary.

So let me work on the other side first. If $\mathscr{C}$ was a group, this would be a principal G-bundle. I can think of these given in terms of coverings and coordinate functions. Let me just give the answer. $[X, B\mathscr{C}_d] = \{(U, \{\varphi_{RS}\}) | U = (U_j)$ a locally finite cover of X and φ_{RR}in $\mathrm{Maps}(\wedge U_j, ob\ \mathscr{C}_d)$, [unintelligible]a certain kind of Steendrod description except done up to equivalences. Let me say very briefly how you may think of moving from one to the other. Both of these are the concordance classes of certain sheafs. If you can prove that your map of concordance classes is one-to one, relative, it tells you it's onto.

So then you need the right bundle. If I take $W = M \times \mathbb{R}$ then I have a map to $X \times \mathbb{R}$. This is a proper map because projection is. I have TM and then the map $TW \to TX$ is surjective. So really putting these two things together, by Phillip's submersion theorem, this surjective map, I can isotope W to $\tilde{W}$ so that $\tilde{W} \to X$ is actually a submersion and I still have $\tilde{W} \to X \times \mathbb{R}$ proper.

Then I get [unintelligible], so how can I find something mapping to $\tilde{W}$? Choose a fine enough covering of X such that we can find regular values v_j for the map $f : \pi^{-1}U_j \to \mathbb{R}$. So this, because it's regular, we have a proper submersion over these regular $\mathbb{R}$s, so this gives us actually a fibration. We can similarly find the other side by taking intervals, and get an element in the other class with the outlined description.

Why do we have to have the full category $\mathscr{C}_d$ instead of $\mathscr{C}_{d,b}$. You lose control during the isotopy. Your manifold might have a regular value chosen which partitions to create no outgoing boundary. You can move this thing to infinity, but you have to be delicate, do it in families.

Okay, thanks. Questions?

[I want you to say the whole proof again, you want to go from the infinite loop side to the bundle side, you apply transversality, apply submersion, and choose a regular value. Can you draw a picture and say those three sentences. Don't say the letters, I hate the letters.]

4 Getzler,

So, my talk is going to rely on supermanifolds, I want to dedicate this talk to Raoul Bott, I hope he'd be amused. I remember in my time at Harvard Raoul trying to figure out what a supermanifold was, but I don't think he ever did. It's a dominant memory.

I need to use this idea as the foundation of my talk, differential graded manifold. Locally a manifold has coordinates, but each coordinate has an integer grading. The coordinates ξ^a have their own degree $\deg(\xi^a) \in \mathbb{Z}$. If we have a polynomial with homogeneous total degree $fg = \pm gf$, this is the Quillen sign convention. This is commutative unless f and g both have odd total degree. I can look at one, or think of a single coordinate chart, and then I can do a lot of standard geometry, I can work with vector fields. One way to say what that is would be a graded derivation of the ring of functions.

It's linear and homogeneous, and again using the Quillen sign convention we have $X(fg) = X(f)g \pm fX(g)$. You may already be able to think of an example, whose ring of functions is just the differential forms on an ordinary manifold. Let M be a manifold with $\Omega^*(M)$ the differential forms on M. This is the ring of functions on a graded manifold, locally the coordinates will be the coordinates on M and d of such coordinates. So these are $\{x^i, dx^i\}$

in degrees zero and one. So the pieces are dually in degree 0 and -1. So $T[1]M$ shifts fibers down in degree by 1.

[The functions on a manifold have a property that makes them a manifold. What is the cognate of this?]

Something about freedom, or maybe a Hochschild homology condition.

X might raise degree by 1, for example the exterior differential d. I'm going to replace dx^i with ξ^i. So $d = \sum_i \xi^i \frac{partial}{\partial x^i}$.

The Lie (graded) bracket of vector fields is $[X, Y] = XY \mp YX$. Check as an exercise that if these are derivations then so is the bracket, that really takes the sign conventions for a spin.

In particular, if X is a derivation of odd degree, then $[X, X] = 2XX = X^2$, a double derivation which also happens to be a derivation. Now a differential on a graded manifold is a derivation of degree one whose square is zero. So $(T[1]M, d)$ is a differential graded manifold.

If you haven't heard about these before, you'll hear about them again, they'll be used more and more, they're important in understanding the Gromov-Witten invariants in terms of moduli space.

I'm using them today for something else, to understand some of the algebra behind string topology. A Poisson manifold is a manifold with a differential operator which is also a Lie bracket. The two classic examples, the oldest example is T^*M, a symplectic manifold which yields a Poisson bracket. The second example is the dual of a Lie algebra $\mathfrak{g}^*$. There's a one to one correspondence between linear Poisson brackets on a vector space and Lie algebra structures on its dual in finite dimension.

A Gerstenhaber manifold is a graded Poisson manifold whose Poisson bracket has degree -1. This is a differential geometer's version of a Gerstenhaber algebra. So the nice thing about this definition is that I also have two examples, maybe slightly less classical. One, just as the ordinary cotangent bundle of M is a Poisson manifold, so the shifted cotangent bundle $T^*[1]M$ is a Gerstenhaber manifold. (Batalin-Vilkovisky) But this goes back further to tho algebra of $\Gamma(M, \wedge^* TM)$, the multifunctions. This was studied by Schaten in 1938. The other thing I'll be interested in is $\mathfrak{g}^*[1]$. The dual of a graded vector space reverses the sign of the grading, then you shift the degree down by one.

Now I want to discuss when you have a differential on such a thing. If N is a Gerstenhaber manifold, (This is not a manifold but a graded manifold, but the same is true for a graded manifold.) and d is a differential it is compatible with the Gerstenhaber bracket if $d\{f, g\} = \{df, g\} \pm \{f, dg\}$. This talk is interested in a certain class of differential Gerstenhaber manifold. A differential on $T^*[1]$ corresponds to a Poisson bracket on M. For $\mathfrak{g}^*$ if $\mathfrak{g}$ is concentrated in degree zero, it corresponds to a slight (homotopy) generalization of a Lie bialgebra structure on $\mathfrak{g}$. If the coefficients of d are homogeneous quadratic, this is exact, otherwise it will be L_∞.

Now I have to talk a bit about L_∞ algebras. One can do it in the language of graded manifolds, but that obscures what they are. This will be puzzling if you haven't seen it before. A differential graded Lie algebra $\mathfrak{g}$ has $d : \mathfrak{g} \to \mathfrak{g}$ and a bracket $[,] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ of degree zero. The L_∞ structure continues this with $[x_1, \ldots, x_k] : \mathfrak{g}^{\otimes k} \to \mathfrak{g}$ of degree $2 - k$.

The relationships look like $\sum \pm [[\ldots] \ldots] = 0$, where these are partitioned over the brackets in all possible ways. For the first one we have $[[x_1]] = 0$, for the second we have $[[x_1, x_2]] \pm [[x_1], x_2] \pm [[x_2], x_1] = 0$, which says that d is a derivation of the bracket. The next one says that we have Jacobi up to a third order term, and so on. I should have said that these are graded skew-symmetric.

This is a Lie analogue of Stasheff's A_∞ algebra, it's a homotopy invariant property. The best place I know to read about this is in Fukaya's lectures from somewhere.

If you have a differential graded Lie algebra then $H^*(\mathfrak{g}, d)$ is an L_∞-algebra with vanishing differential.

I'll be interested in the opposite, such as in string topology, where you have a Gerstenhaber structure at the homology level, so an L_∞ structure at the chain level.

All of this is characteristic zero, that's a caveat.

I want to explain now the homotopy analogue of a Gerstenhaber structure. Now suppose that G^* is a graded vector space. I'll define the notion of a G_∞ algebra structure on G^*. This will be a lot of different multilinear products. If all vanish but the bilinear one, we'll get a Gerstenhaber algebra. This takes the bilinear products fg and $\{f, g\}$ on a Gerstenhaber algebra and augments them with a differential and an enormous list of higher operators. We know that we're interested in this because it will be the cohomology version of a differential Gerstenhaber algebra.

Let $\mathfrak{g}$ be a graded vector spac. Then $\mathfrak{g}[1]$ is a graded manifold, with functions $\wedge^*(\mathfrak{g}^*)$. Then an L_∞ structure on $\mathfrak{g}$ is the same thing as a differential on this graded manifold.

This is relatively well-known. I want to accept this as a model for how to do this sort of thing. So $d^2 = 0$ is higher associativity and the coefficients are the higher operations.

[You could discuss this from other points of view, you don't need to mention differential graded manifolds.]

I never need to mention them, I think this is a nice way of looking at this.

A differential d on the Gerstenhaber manifold $Lie(G[1]^*)^*[1]$ is a G_∞ structure. This is a reformulation of a theorem I proved with John Jones. This is Tamakin's reformulation. In particular inside is a shifted copy of G, so there's an L_∞ structure kicking around.

Why am I interested in this differential geometry style language? I noticed that the notion of the differential is ugly, it's a derivation of the bracket which is hard to manipulate.

You want to observe that this is actually a Batalin-Vilkoviski manifold, a BV-manifold.

A BV-manifold is a Gerstenhaber manifold together with a second order differential operator Δ of degree -1 such that

1. $\Delta^2 = 0$
2. $\Delta(fg) = \Delta(f)g \pm f\Delta g + \{f, g\}$ up to some signs.
3. $\Delta 1 = 0$.

Example 1 *If we take $\mathfrak{g}^*[1]$ and a homogeneous basis e_n of $\mathfrak{g}$, then Δ is given by, well,*

$$\frac{1}{2}\sum_{i,j,k} C^i_{jk} e_i \frac{\partial^2}{\partial e_j \partial e_k}$$

This is the differential which is not a derivation which defines a Lie algebra homology. This correlates the concepts here with some we're familiar with. Now I can state my theorem.

Let N be a Gerstenhaber manifold with BV operator Δ. Then a differential on N (for the Gerstenhaber structure) is a differential on the graded manifold underlying N (a graded derivation with respect to the product) such that the anticommutator $\Delta d + d\Delta$ is a derivation (i.e., vector field). It will be in degree zero. I think this is a beautiful reformulation, and secondly focuses attention on this vector field.

In the few minutes remaining let me talk about topological gravity and Frobenius manifolds.

Suppose I have a topological field theory H in dimension two. I mean something a little more refined than Atiyah. If H is graded we have Bor_2, a simplicial category, where morphisms between objects are simplicial sets.

Now Vec is also a simplicial category. If I have H_0 and H_1 then the simplicial maps are $Vec_{\cdot}(H_0, H_1) - Hom(V_0, \Omega^*(\Delta^{\cdot}) \otimes V_1)$. This is the proper notion in the graded case.

There is a theorem that H will then be a G_∞-algebra. No more questions. So Tamakin essentially proves that H is a G_∞ algebra. We expect that when we couple to topological gravity, we get a Frobenius manifold.

Let H be Hochschild cochains on the Fukaya category of a compact K ahler manifold. Then we expect a Frobenius manifold somewhere (solution of the WDVV equation).

To finish I want to say what a $WDVV_\infty$ algebra is, that's a G_∞-algebra with $\Delta d - d\Delta = 0$.

The big meat in the sandwich in Costello, Kontsevich in constructing Gromov-Witten invariants is that the Hochschild cochains on the Fukaya category of a compact K ahler manifold are $WDVV_\infty$.

So let me say for the WDVV potential $\Delta\eta^{ab}\partial_b\Phi) = dt^a$. This completes the circle.

A BV_∞ algebra have an auxilliary parameter u of defree two. So $d = d_0 + ud_1 + u^2 d_2$ and you have $(d + u\Delta)^2 = 0$. The linear piece of d_1 is the BV operator.

Now we have a one minute break.

5 DeCoto, K-theory cohomology and elliptic [unintelligible]

Okay, so here we have two cohomology theories, H^* (ordinary cohomology) and K, K-theory. There is a third one $\mathscr{E}$, elliptic cohomology, which has some nice features in common. They have geometric models, many applications, including some relations with physics, and of course many other applications. I will take the point of view that the second nice feature is a consequence of the first one. So deRham cohomology, vector bundles, and elliptic objects. I will show some evidence that there is another nice theory here called $K3$-cohomology. I had the hope that everyone knew what elliptic cohomology was, but it seems not.

These theories are related to formal group laws. For ordinary cohomology, $x+y$, for K-theory the multiplicative group law $x+y+xy$, for elliptic cohomology tho formal group law af an elliptic curve.

Suppose you have a Lie Group G, with identity e. Suppose dim $G = 1$, then you can pick coordinates. At each point p you can pick $Z(p)$. Choose $Z(e) = 0$. Then $Z(p+q) = F(z(p), z(q))$ where this is a formal group law.

You can have formal group laws without having a group. When is this geometric, when does it come from a group? The three I've mentioned are the only ones. Let me go back to the case of the elliptic curve. You can obtain the formal group law as I described here. Suppose you have an elliptic curve F over *spec* k. Take [unintelligible]over a local Artinian algebra, for example $A = k[\epsilon]/\epsilon^2 = 0$, with $A \to k$ (killing nilpotents).

Then you get

$$\begin{array}{ccc} E & \longrightarrow & E \times_k A \\ \downarrow & & \downarrow \\ \textit{spec } k & \longrightarrow & \textit{spec } A \end{array}$$

You get $A \to \ker\{H^1(E \times_R A, G_m) \to H^1(E, G_m)\}$, the tangent space to the Picard group. Then $A \to Pic(A)$. Now you can take other varieties and cohomology in different degrees.

So now let X be a variety over A, replacing E with it:

$$\begin{array}{ccc} X & \longrightarrow & X \times_k A \\ \downarrow & & \downarrow \\ \textit{spec } k & \longrightarrow & \textit{spec } A \end{array}$$

and then you get $A \to \ker\{H^2(X \times_R A, G_m) \to H^2(X, G_m)\}$, the tangent space to the

Brouwer group of X. So you send A to $\mathrm{Br}(A)$.

So $H^{1,0}(X) = H^{3,0}(X) = 0$. and dim $Br(X) = \dim H^{2,0}(X)$. We need varieties of this kind. So let X be $K3$-surfaces. Of course you have a formal group but you don't have a formal group law. You need a nice set of equations, and then you want to work with the whole moduli of $K3$-surfaces, of which there are no nice descriptions, and you need relative deformation theory, very serious problems of algebraic geometry.

I will take another tack, I want to look for a candidate for the geometric background.

Recall $\mathscr{E}(X) = K(\mathscr{L}(X))$. CFTs for Segal begin with closed 1-manifolds as objects and for morphisms Riemann surfaces. Here there are two cases that are important, the first case is a certain semigroup, that formed by surfaces which are topologically annuli. This appears as the complexification of $Diff_+(S^1)$.

Now what can we do from the point of view of $K3$-cohomology? Well, there are two different ways of looking at things. The first thing is to say that $K3$-homology is the next step, so we could dream that something like this would be true: $K3(X) = \mathscr{E}(\mathscr{L}X) = K(\mathscr{L}\mathscr{L}X)$.

[Why is that $K3$ and not a $T4$?]

This gives the right dimension, with that you will get 2-dimensional formal group laws.

Here i should have 2-tori instead of circles, with morphisms which are 3-manifolds. Now there is an obvious problem here. In the old case you glued your annuli and get an elliptic curve. Here you will get 3-manifolds, where you want 4-manifolds.

Instead of taking 2-tori, I'll take (T^3, Ω), sorry Dennis, I don't know how to draw 3-tori.

[Cube. Identify the sides.]

Donaldson, Nahm

Definition 2 *Suppose you take G a Lie group that acts on a real space V, then we say that $U^+, U^- \in V \otimes \mathbb{C}$ are $G^{\mathbb{C}}$-related if*

1. $g \in G$
2. *$U_1(t), U_2(t)$ are smooth families parameterized by $t \in [-a, a]$ such that $U_1(-a) + iU_2(-a) = U^-$, $U_1(a) + iU_2(a) = gU^+$.*
3. *$\alpha(t) \in \mathfrak{g}$ such that $\frac{d}{dt}(U_1 + iU_2) = i\alpha(t)(U_1 + iU_2)$.*

This is Donaldson's approach. Now Nahm's approach explains how you can, well, Nahm's equation has $\mathfrak{g}$ a Lie algebra and $g_1(t),\ g_2(t),\ g_3(t)$. Then you introduce $\frac{dg_i}{dt} = [g_j, g_k]$ whenever ijk is a cyclic permutation of 123. Then you take $\frac{d}{dt}(g_1 + ig_2) = [ig_3, g_1 + ig_2]$

You get $U_1(t) = g_1(t),\ U_2(t) = g_2(t)$ when V is the adjoint representation of $\mathfrak{g}$.

This was moved to a geometric setting by considering $T^3 \times [-a, a]$, called Σ.

Definition 3 *A complex symplectic surface Σ is a 2-dimensional complex manifold with a holomorphic symplectic form $\theta \in \Omega^{2,0}$.*

This notion encodes the solution of Nahm's equation and [unintelligible]. So the elements you will take in order to get Nahm's element, $G = \mathrm{Diff}_\Omega(T^3)$ (that preserve the volume form). This gives three 1-parameter vector fields $v_1(t), v_2(t), v_3(t)$ that satisfy Nahm's equation. Now from this data Donaldson shows that you can define θ. He conjectures that you can go the other direction. So this helps with some sort of complexification.

Now if you use the three different choices you get three different structures $\theta_1, \theta_2, \theta_3$. This is because this is really a quaternification, since Σ is hyperK ahler.

Someone else showed in another way that this is a quaternionification. Here you need three elements. With time, you get the real part. Making a gauge transformation you can get rid of the real part and see that the important part is what is left over.

The next thing to do is to get a representation.

[Are you saying that for any three vector fields you can do something?]

You need them to be divergence-free and [unintelligible].

Now can we do any representation theory?

[Here my notes become fragments.]

$Sp(1) = \not{D} = i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$.
$\mathscr{S}(\mathbb{R}^3, \mathbb{H}),\ e^{-t\not{D}}$
$L^2(T^3, \mathbb{H}) = H^+ \oplus H^-$.

Of course the problem is that the tensor product is very hard to use. To do many things you can replace the complex numbers with quaternions. I have heard many times in the last few years, a $K3$-surface is the quaternionic version of an elliptic curve. I don't understand this, but maybe one of you can explain this to me. You want to get some sort of Abelian manifold with an involution that you divide by, blow up and you get a K uhmer surface.

That's all.

[Do you have physics background after $K3$-cohomology.]

These came from the work of [unintelligible]on quantum gravity.

6 Westerland, Equivariant operads and string homology

M will be a closed d-manifold and LM will be $\mathrm{Maps}(S^1, M)$ I probably mean piecewise smooth, then S^1 acts on LM by rotating loops and you can form the Borel construction which I will denote $LM_{hS^1} = ES^1 \times_{S^1} LM$.

Theorem 3 *(Chas, Sullivan) There exists an uncountable number of Lie_∞ structures on $H_*(LM_{hS^1})$.*

I guess the goals for today are

1. Explain this in a stable homotopy context
2. Adapt the idea to more general equivariant settings.

I haven't said what this means.

Let's talk about the Lie_∞ or gravity operad (Getzler, Ginzburg-Kapranov).

Definition 4 *Let $\mathscr{M}_{0,n}$ be the moduli of n points in $\mathbb{CP}$. Then* $\mathrm{Grav}(n) = \Sigma H_*(M_{0,n+1})$.

Theorem 4 *(Getzler)*
$\{\mathrm{Grav}(n)\}$ *forms an operad, called the gravity operad or the Lie_∞ operad.*

For the substitution, use the operad structure on $H_*(\bar{M}_{0,n+1})$ and get back to the other by using Poincaré residue maps. This is slightly mysterious, later I'll give another way.

Here's a presentation. Grav is generated by $\{a_1, \ldots, a_n\} \in \mathrm{Grav}_n$ of degree one, subject to a generalized Jacobi relation, similar to what Ezra wrote down earlier. It's a mess so I won't write it out. Today we'll give an alternate construction with homotopy theory.

A restatement of the Chas-Sullivan theorem is $\Sigma^{1-d} H_*(LM_{hS^1})$ is an algebra over Grav and $\{a, b\}$ is the string bracket.

More generally, $\{a_1, \ldots, a_k\} = \pi(\tau(a_1) \bullet \cdots \bullet \tau(a_k))$ where τ is the S^1 transfer to $H_*(LM)$ and $\bullet$ is the loop product, with π the projection back to $H_*(LM_{hS^1})$.

This is a restatement of the loop space as a gravity algebra, which may make you think this will be a physics talk, but it's going to equivariant homology.

[Why is it called gravity?]

[Some reasons in response.]

[I call it a poly-Lie algebra.]

[Gabriel showed me something in Manin, he seems to call it Lie_∞.]

[I don't think so.]

So that takes me to this more general setting,

6.1 General Equivariant Setting

Definition 5 *(Salvatore, Wahl)*
If G is a group, a G-operad (in spaces) is an operad $\mathscr{C}$ whose substitution maps $\mathscr{C}(k) \times \mathscr{C}(n_1) \times \cdots \times \mathscr{C}(n_k) \to \mathscr{C}(\sum n_i)$. are G-equivariant. It's an operad in the symmetric monoidal category of G-spaces.

The little disks operads $\mathscr{C}_n$ is an $SO(n)$ operad.

Definition 6 *A naive G-operad is an aperad in the category of naive G-spectra, where spectra are S-modules. (EKMM)*

Jim said that operads can be a bit intimidating to the beginner, I hope this doesn't help.

An example is $\Sigma^\infty \mathscr{C}_+$ where $\mathscr{C}$ is a G-operad. The spaces in the spectrum are G-spaces, and the maps are equivariant, and the group does not act on the suspension.

So here's a fact I'm going to use throughout. If C is compact, Lie, then the transfer map ${}_\tau G : EG_+ \wedge_G (S^{Ad_G} \wedge Y) \to (EG_+ \wedge Y)^G$ is an equivalence for all naive G-spectra Y. This is just a twisted suspension of the Borel construction. Take the Lie algebra of the group, then the one-point compactification, the group acts on it by conjugation, and so on.

Theorem 5 *(W.)*
If G is a naive G-operad then $(EG_+ \wedge \mathscr{C}(k))^G$ form a (non-unital) operad in the category of spectra.

An operad is only as good as its algebra.

[You're only going to apply the transfer, right?]

No, that's not strict.

[That's why it's a theorem, I guess.]

Theorem 6 *If X is an algebra over $\mathscr{C}$ in the category of naive G-spectra (equivalently, X is a $\mathscr{C} \rtimes G$-algebra), then $(EG_+ \wedge X)^G$ is an $(EG_+ \wedge \mathscr{C})^G$ algebra.*

Example 2

S^1 acting on $\mathscr{C}_2 \rightsquigarrow) ES^1_+ \wedge \mathscr{C}_2)^{S^1}$ where $ES^1_+ \wedge \mathscr{C}_2(k)_+)^{S^1} \cong ES^1_+ \wedge_{S^1} (S^{Ad_{S^1}} \wedge \mathscr{C}_2(k)_+) = \Sigma ES^1_+ \wedge_{S^1} \mathscr{C}(k) = \Sigma\Sigma^\infty(\mathscr{C}(k)_{hS^1})$.

Note $k = 1$ has $\mathscr{C}_2(1)$ a point over $S^1 2$ so $\mathscr{C}(1)_{hS^1} \cong BS^1$. For $k > 1$ the action is free, $\mathscr{C}(k) \cong_{S^1} F(\mathbb{C}, k)$ and $\mathscr{C}(k)_{hS^1} \cong F(\mathbb{C}, k)/S^1 \cong M_{0,k+1}$.

Corollary 2 $(H_*((ES^1_+ wedge \mathscr{C}_2)^{S^1}))_{>1} \cong \mathrm{Grav}$.

We can study $\mathscr{C}_n$ with the $SO(n)$ action, which is a mess. What is maybe more interesting, is to look at $(ESU(2)_+ \wedge \mathscr{C}_4)^{SU(2)}$. If I take the homology operad of this guy, throwing away the first degree, it is also the gravity operad: $(H_*(ESU(2)_+ \wedge \mathscr{C}_4)^{SU(2)})_{>1} \cong \mathrm{Grav}$ (degree shearing).

So this makes one think 4 dimensional quantum gravity might be related to 2 dimensions, they at least smell the same.

[In the parlance of the last talk this is a quaternionic version.]

Yes, it's like a quaternionic moduli space.

6.2 Back to string topology

Theorem 7 *Cohen, Jones*
$\Sigma^{-d} H_*(LM)$ is an algebra over $H_*(\mathrm{Cacti})$ (BV-algebra), the cacti being $\cong \mathscr{C} \rtimes S^1$.

We want to use the previous theorem to give a gravity algebra structure on $H_*(LM_{hS^1})$. That doesn't work for a number of reasons. There's not actually a framed little disks action. There are appropriate maps that let you do it in homology for any homology theory where M is [unintelligible].

So problems are,

- LM^{-TM} is not a $\mathscr{C}_2 \rtimes S^1$-algebra.
- The cactus operad has a suboperad corresponding to the unframed little disks operad:

$$\begin{tikzcd} \mathrm{Cacti} \arrow[d, leftrightarrow] & \mathrm{Cact} \arrow[l, hook'] \arrow[d, leftrightarrow] \\ \mathscr{C}_2 \rtimes S^1 & \mathscr{C}_2 \arrow[l, hook'] \end{tikzcd}$$

  But Cact is not an S^1-operad, where $\mathscr{C}_2$ is. Cacti is a bicrossed product of Cact and S^1.

I'm over time, so thank you.

[[unintelligible]couple the theory with topological gravity, a choice of an element in $M_{0,n}$.]

That's the next direction to go. This relates to something Gabriel was saying to me, about creating a Taylor function that is like a partition function, if I understand you right.

We're not picking a point in the moduli space, we're taking the whole thing.

Other questions? Thank you again. We'll see you tomorrow.