Stringy Topology Notes
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1 Godin, String Topology

So first I'll recall what we did last time, I'll try to give more of the Chas-Sullivan construction, and then go to what others have done.

The best reference for a broad overview is Cohen-Voranov, "Notes on String Topology."

Recall that last time wo set LM to be $\mathrm{Maps}(S^1, M)$ with M a smooth oriented map. The maps are piecewise smooth. Using this we built a loop product on the homology $\bullet : H_pLM \otimes H_qLM \to H_{p+q-d}LM$.

So first, just to get rid of the dimension change, let $\mathbb{H}_* = H_{*+d}LM$ Then $\bullet : \mathbb{H}_p \otimes \mathbb{H}_q \to \mathbb{H}_{p+q}$. So now I'll build a delta and a bracket to make this a BV-algebra, and then move on to work since Chas-Sullivan.

So first I'll define the loop bracket. Let me recall the $\star$ operator which we used to prove $\bullet$ was commutative. We consider $\mathscr{P} = \{\alpha, \beta, t | \alpha(0) = \beta(t)\} \overset{\rho}{\subset} LM \times LM \times I$.

So if you do this for $t = 0$ you get the loop product back, but if we do this one dimension higher, well, recall $\star : C_pLM \otimes C_qLM \overset{\otimes[I]}{\to} C_pLM \otimes C_qLM \otimes C_1 I \to C_{p+q+1}LM \times LM \times I \overset{\rho!}{\to} C_{p+q+1-d}(\mathscr{P}) \overset{\text{comp}}{\to} C_{p+q+1-d}LM$. This was used to show commutativity of $\bullet$.

Define the loop bracket as follows: $\{x, y\} = x \star y \pm y \star x$, and as always there is a sign I won't bother with.

Lemma 1 *This gives a map in homology of degree one* $\{,\} : \mathbb{H}_p \otimes \mathbb{H}_q \to \mathbb{H}_{p+q+1}$

From the last time $\delta(x \star y)$ is $\delta x \star y + x \star \delta y \pm x \bullet y \pm y \bullet x$. So if x and y are cycles, what you get is $\delta(x \star y \pm y \star x) = x \bullet y \pm y \bullet x \pm x \bullet y \pm y \bullet x$ and the signs are going to cancel and you get 0. So this bracket actually gives a Gerstenhaber algebra.

Theorem 1 $(\mathbb{H}_*, \bullet, \{,\})$ *is a Gerstenhaber algebra, namely*

1. *$\bullet$ is associative and graded commutative, which we showed last time.*
2. *$\{,\}$ is a Lie bracket of degree one with appropriate compatibility, which means:*
   (a) *$\{a,b\} = \pm\{b,a\}$, which is clear from the definition.*
   (b) *$\{a,\{b,c\}\} = \{\{a,b\},c\} \pm \{b,\{a,c\}\}$.*
   (c) *$\{a, b\bullet c\} = \{a,b\}\bullet c \pm b\bullet\{a,c\}$, the bracket is a derivation of the product.*

Proving Jacobi is similar to what we've done. We'll prove that bracket is a derivation of the product. So we need $x \star (y_1 \bullet y_2) = (x \star y_1)\bullet y_2 + y_1 \bullet (x \star y_2)$. The proof here is by picture again. If x is at some point on the product of loops, either it's on one loop (before 1/2) or on the other loop.

The second thing we need is that $(x_1 \bullet x_2) \star y = x_1 \bullet (x_2 \star y) \pm (x_1 \star y)\bullet x_2$. This one is not true on the chain level, it's chain homotopic. I'll construct the homotopy. We want to take y with x_1 and x_2 on it at some point. You can move them together to get the left side. If $\mathscr{P} = \{(\alpha,\beta,\gamma,s,t), s \le t, \gamma(s) = \beta(0), \gamma(t) = \alpha(0)\} \overset{\rho}{\subset} LM \times LM \times LM \times \Delta^2$. If we do $\rho!(x_1 \otimes x_2 \otimes y \otimes [\Delta^2])$ we get $C_*(LM^3 \times \Delta^2) \to C_*(P) \to C_*(LM)$. So this thing lives over Δ^2. where one edge corresponds to $(x_1 \star y)\bullet x_2$, another corresponds to $x_1 \bullet (x_2 \star y)$, and the third edge corresponds to $(x_1 \bullet x_2) \star y$.

This tells you that the sum of these with some orientations is zero in homology.

Now we have an S^1 action which will give us a new operator, the Δ-operator. The S^1-action will be given by $\psi : S^1 \times LM \to LM$, $\psi(t,\alpha)(s) = \alpha(s+t)$. Every time you have an S^1 action you can do the following thing, we define $\Delta : C_*LM \overset{\times[S^1]}{\to} C_{*+1}(LM \times S^1) \overset{\psi}{\to} C_{*+1}LM$ which induces $\Delta : \mathbb{H}_* \to \mathbb{H}_{*+1}$. So $\Delta^2 = 0$ since $\Delta^k(\alpha)$ for $k \ge 3$ has the same geometric image as $\Delta(\alpha)$.

Proposition 1 *The loop bracket $\{,\}$ is the deviation of Δ from being a derivation of the loop product $\bullet$.*

This means that $\Delta(a\bullet b) \pm \Delta(a)\bullet b \pm a \bullet \Delta(b) = \{a,b\}$ instead of zero.

Theorem 2 $(\mathbb{H}, \bullet, \Delta)$ *is a BV-algebra.*

1. *$\bullet$ is associative and graded commutative.*
2. $\Delta^2 = 0$
3. *$\Delta(a\bullet b) \pm \Delta(a)\bullet b \pm a\bullet\Delta(b)$ is a derivation in both variables.*

This is all I'm going to talk about from this paper, you want to mod out by the circle action to get these to be strings instead of loops.

Now, how much does smoothness matter? This is work in a recent paper of Cohen-Klein-Sullivan. It's from the arXiv around September.

Theorem 3 *Say you have a map $f : M_1 \overset{\cong}{\to} M_2$. Say M_1 and M_2 have the same dimension and say f preserves the orientation class in homology, then $f_* : H_*LM_1 \overset{\cong}{\to} H_*LM_2$ is an isomorphism of loop algebras, meaning there is an isomorphism of the loop product, and on the bracket in S^1-equivariant. The bracket can be described using Δ and $\bullet$. The S^1 action on both sides must be compatible. For some reason they don't seem to think the bracket is preserved.*

They look at configuration spaces and prove them invariant up to homotopy, this is a bit weird because a lot depends on the smooth structure. This was motivated by

Theorem 4 *Cohen-Jones If M is simply connected there is an isomorphism between $H_{*+d} \overset{\cong}{\to} HH^*(C^*(M), C^*(LM))$ (what is preserved is the product)*

[Some discussion, how do configuration spaces relate?]

Points outside the diagonal.

1.1 Stable homotopy point of view

The point is to lift the construction on the chain level to a construction on spectra. You can Twist the Thom collapse map; if you have an embedding of two smooth manifolds and a bundle

$$\begin{tikzcd} & \zeta \arrow[d] \\ N \arrow[r, hook, "e"'] & M \end{tikzcd}$$

and you have a section you get $N \hookrightarrow \zeta$, the normal bundle of $f \cong \nu_e \oplus e^*(\zeta)$.

You get

$$\zeta \to \mathrm{Thom}\begin{pmatrix} e^*\zeta \oplus \nu_e \\ \downarrow \\ N \end{pmatrix}$$

which induces $\mathrm{Thom}(\zeta) \to \mathrm{Thom}(e^*\zeta \oplus \nu_e)$.

Remark, If $\zeta = \gamma_1 - \gamma_2$ is a virtual bundle, instead of a Thom space you get a Thom spectra, so $\zeta \oplus \xi_n$ (a trivial bundle) you get an actual bundle so $\mathrm{Thom}(\zeta \oplus \xi_n)$ exists. In general $Thom(\xi_n) \cong M \times D^n / M \times S^{n-1} = M \times S^n / M \times \{*\} = \Sigma^n M$ so we define $\mathrm{Thom}(zeta) = \Sigma^{-n}\mathrm{Thom}(zeta \oplus \xi_n)$.

So consider

$$\begin{tikzcd} ev^*(-TM) \arrow[r] \arrow[d] & -TM \arrow[d] \\ TM \arrow[r, "ev"] & M \end{tikzcd}$$

We get a diagram

$$\begin{tikzcd} ev^*(-2TM) \arrow[r, hook] \arrow[d] & ev^*(-TM) \times ev^*(-TM) \arrow[d] \\ \mathrm{Maps}(\infty, M) \arrow[r, hook, "\rho"] & LM \times LM \end{tikzcd}$$

We get

$$\begin{tikzcd} \mathrm{Thom}(ev^*(-TM) \times ev^*(-TM) \cong \mathrm{Thom}(ev^*(TM)) \wedge \mathrm{Thom}(ev^*(-TM)) \arrow[d] \\ \mathrm{Thom}(ev^*(-2TM) \oplus ev^*(TM)) \cong \mathrm{Thom}(ev^*(-TM)) \end{tikzcd}$$

So we get $\mathrm{Thom}(ev^* - TM)^{\wedge^2} \to \mathrm{Thom}(ev^* - TM)$.

Theorem 5 *(Cohen-Jones) Gor any closed smooth manifold* $\mathrm{Thom}(ev^*(-TM))$ *is a ring spectrum.*

If M is oriented then $H_*(\mathrm{Thom}(ev^* - TM)) \cong \mathbb{H}_*$ by the Thom isomorphism is an isomorphism of graded algebras.

[Discussion about whether the framed or unframed cactus operad acts on something.]

2 Kitchloo