Stringy Topology Notes
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How many of you had to take a taxi this morning? If you are just arriving, welcome, please register with our secretary, should we start?

1 Kitchloo, Loop Groups

Thank you, I've been having a great time, thanks for having me, the lectures have been very clear, unfortunately I'm not going to be able to do that. It's slightly technical.

A reference, Loop Groups by Pressley and Segal.

What are loop groups? Let me remind you of a compact Lie group. For example, let $K = U(n)$, the endomorphisms of a complex vector space that preserve the Hermitian inner product. In general a compact Lie group is a compact subgroup of $U(n)$. This is not uniquely defined, but for what I'm doing today assume that this embedding is defined. Today assume K is connected.

Definition 1 *The smooth loop group $L_S(K) = C^\infty(S^1, K)$. The group structure is pointwise multiplication.*

The algebraic loop group $L_a(K) = \{A(z) \in L_S(K) | A(z) = \sum_{-m}^{m} A_i z^i\}$ where $A_i \in End(\mathbb{C}^n)$.

A remark, $L_a(K)$ is a group because the inverse is the conjugate.

Remarks on the topology:
These are actually topological groups. On $L_S(K)$ the topology will be that of uniform convergence of all derivatives. $f_k(z) \to f(z)$ in $L_S(K)$ if and only if $\frac{d^s}{dz^s} f_k(z) \to \frac{d^s f(z)}{dz^s}$ uniformly on S^1 for all s.

Now let me define the topology for the algebraic loop group. Let $L_a^m K = \{A(z) \in L_a K | A_i = 0 \text{ if } |i| > m\}$. Then $L_a^m K$ can be given the induced topology from L_S^K which makes it a compact subspace, and then give $L_a(K)$ the direct limit topology. So $T \subset L_aK$ is closed if and only if $T \cap L_a^m K$ is closed for all m.

Note that the obvious inclusion is continuous. So I just showed these to be topological groups and showed their relation. Let me talk about Lie algebras. If I have a nice topological group, the tangent space at the identity has a Lie algebra.

So L_aK and L_SK have corresponding Lie algebras. Let $\mathfrak{g}_k$ be the Lie algebra of K; then the Lie algebra of L_SK is $C^\infty(S^1, \mathfrak{g}_k)$. I will discuss L_aK next time. The bracket is the obvious induced one. These each have corresponding exponential maps, which in the case of L_SK is a local homeomorphism. For L_aK this is defined only on a dense set, the exponential will be a power series, not a polynomial. But if you pick a nice element it will be truncated.

So, this is good, I am going as fast as I planned to go. Let me state a theorem here, the important theorem in this situation.

Theorem 1 *(Garland, Quillen, Ragunathan)*
If K is semisimple (e.g., if $\pi_1(K) = 0$) then the map $L_aK \to L_SK$ is dense and it is a homotopy equivalence.

These are highly nontrivial statement. If K were a torus, a circle, the algebraic group is the circle times powers. The diffeomorphisms are humongous. That it is a homotopy equivalence is another powerful statement. We do this because the algebraic loop groups are easy to study, but we want to study the smooth one, since they are homotopy equivalent we study the algebraic one.

1.1 The Grassmann model for $L_aU(n)$

Fix $K = U(n)$.

Recall that $L_aU(n) \subset End(\mathbb{C}^n)[z, z^{-1}] \subset End(\mathbb{C}^n[z, z^{-1}])$ where this last is just $\mathbb{C}^n \otimes \mathbb{C}[z, z^{-1}]$.

So in particular $L_aU(n)$ can be thought of as operators on $\mathbb{C}^n[z, z^{-1}]$.

Let's add some structure, starting with an inner product.

We define an inner product on $\mathbb{C}^n[z, z^{-1}]$ by claiming that $e^i \otimes z^j$ for $i \leq n, j \in \mathbb{Z}$ are orthonormal. Let H be defined as $\mathbb{C}^n[z, z^{-1}]$ with $H_+ = \mathbb{C}^n[z], H_- = \mathbb{C}^n[z^{-1}]$.

Let me define more objects of interest.

Definition 2 *Let $Gr = \{W \subset H | z^\gamma H_+ \subset V \subset z^{-\gamma} H_+\}$ (W is a subspace).*

I claim that Gr is homeomorphic to $\mathbb{Z} \otimes BU$. What I mean by this is that there is a canonical model where this is true.

Let me prove this here, a more or less rigorous proof modulo some easy details. Let $H_k = H/z^k H_+ = \mathbb{C}^n[z, z^{-1}]/z^k\mathbb{C}^n[z]$. So one has projection maps $\pi_k : H_{k+1} \twoheadrightarrow H_k$.

Let $Gr(m, H_k) = \{W \subset H_k | \dim\ W = m\} \cong BU(m)$.

So the way these are filters, the quotients are n-dimensional.

[There is an action of S^1 here?] I will not need it but this is S^1-equivariant.

Where were we? Like I said, the kernel of this surjection is n-dimensional, $z^k\mathbb{C}^n$.

So we have induced maps $\pi_k^* : Gr(m, H_k) \to Gr(m+n, H_{k+1})$ which correspond to $BU(m) \hookrightarrow BU(m+n)$.

We have these finite Grassmannians and the maps taking one to another. So we get a map $\amalg_{k,m} G(m, H_k) \twoheadrightarrow Gr$. This sends $W \to W$. In one case it sits inside $Gr(m, H/z^r H_+)$ and in the other case in H.

It's quite clear that this map is surjective and is a well-defined map. I will rearrange this to make it slightly more appealling, we get

$$\varphi : \amalg_{\gamma\in\mathbb{Z}}(\amalg_k gr(\gamma + kn, H_k)) \to Gr$$

It is easy to see that φ factors through the identification induced by π_k^* so it descends to a map

$$\amalg_{\gamma\in\mathbb{Z}}(\amalg_k(Gr(\gamma + kn) \sim Gr(\gamma + (k + 1)n) \sim \cdots)) \to Gr$$

or

$$\amalg_{\gamma\in\mathbb{Z}}(\amalg_k(BU(\gamma + kn) \sim BU(\gamma + (k + 1)n) \sim \cdots)) \to Gr.$$

This gives $\varphi : \amalg_\gamma(BU) \to Gr$ and completes the proof.

Note the map $I : Gr \to \mathbb{Z}$ called the index is given by $I(W) = \dim(W/z^r H_+) - \gamma n$. This number is well defined, independent of γ.

Definition 3 *Let* $\mathscr{L} = \{W \in Gr|zW \subset W\}$.

This is the space of lattices.

Fact: $L_aU(n)$ acts on $\mathscr{L}$. In fact it acts transitively! Moreover, $H_+ \in \mathscr{L}$ and the stabilizer of this group action on H_+ is $U(n) \subset L_aU(n)$ as constant loops.

This gives us that, so we get an inclusion

$$L_aU(n)/U(n) \cong \mathscr{L} \subset Gr = \mathbb{Z} \times BU.$$

Note that $L_aU(n) = \Omega_aU_n \rtimes U(n)$ where $\Omega_aU(n) = \{A(z) \in L_aU(n)|A(1) = id\}$.

That is we get an inclusion $\Omega_aU(n) \subset \mathbb{Z} \times BU$.

1.2 A CW decompositon of $\Omega_a U(n) = (\mathscr{L})$

Recall $Gr = \{W \subset H | z^\gamma H^+ \subset W \subset z^{-\gamma} H_+\}$ and $\mathscr{L} = \{W \in Gr | zW \subset W\}$.

Filter H by subspaces $H_{p,i}$ with $p \in \mathbb{Z}, 1 \leq i \leq n$. So $H_{p,i} = \{\langle e_1 z^{k_1}, e_2 z^{k_2}, \dots e_n z^{k_n} | k_j \geq p$ for $j \leq i, k_j > p$ for $j > i\rangle\}$ Note that $H_{p_1,i_1} \subset H_{p_2,i_2}$ if $(p_1, i_1) \leq (p_2, i_2)$ in the lexicographical ordering.

So given, remember we want a CW decomposition for $\mathscr{L}$. Given $W \in \mathscr{L}$ let $\bar{W}$ be the orthogonal complement of zW in W. I may as well think of W as H_+ and zW as zH_+ since I have that transitive action. Then $\dim \bar{W} = n$. So $\bar{W}$ has exactly n increasing steps $(p_1, i_1), \dots, (p_n, i_n)$.

What I mean is that the intersection of one of these guys with $\bar{W}$ will increase each time by one, it will be zero for a while, then increase by one to n.

Let $\mathscr{W}_{\underline{a}} = \{W \in \mathscr{L} | (p_1, \dots, p_n) = \underline{a}\}$ (it is a fixed multiindex. For example, if $\mathbb{C}[z]\langle e_1 z^{a_1}, \dots, e_n z^{a_n}\rangle = W \in \mathscr{W}_{\underline{a}}$.

Fact, please don't ask me the proof, $\mathscr{W}_{\underline{a}}$ is an affine cell (homeomorphic to $\mathbb{C}^?$) of complex dimension

$$d(\underline{a}) = (\sum_{i<j} |a_i - a_j|) - \#\{(i,j), i < j, a_i > a_j\}$$

Note that $I(x) = \sum_{i=1}^n a_i$ for all $x \in \mathscr{W}_{\underline{a}}, I : Gr \to \mathbb{Z}$ the index.

Consider $\mathscr{L}_0 = I^{-1}(0) \cap \mathscr{L}$. So the previous board gives a CW decomposition of $\mathscr{L}_0$ with cells only in even dimensions. Thus the cohomology is the same as the cochains. So the Poincaré series of $\mathscr{L}_0$, that is $\sum_0^\infty t^i \dim(H^i(\mathscr{L}_0, \mathbb{Z}))$ is given by $\sum_{\underline{a}, \sum a_i = 0} t^{2d(\underline{a})} = \prod_{i=1}^{n-1} (1 - t^{2i})^{-1}$. There is nothing mathematical here besides the combinatorics, that is not how I want to say that.

So I am done, in a minute I'll be completely done. So the Poincaré series of $\Omega_a U(n)$ is $\prod_{i=1}^{n-1} (1 - t^{2i})^{-1}$. Recall $\Omega_a U(n) \subset \mathbb{Z} \times BU$ as a subcomplex. The Poincaré series of BU is $\prod_{i=1}^{\infty} (1 - t^{2i})^{-1}$. Since $H^*(BU, \mathbb{Z}) = \mathbb{Z}[c_1, c_, \dots], |c_i| = 2$.

So $\Omega_a U(n) \subset \mathbb{Z} \times BU$ is a (co)homology equivalence up to degree $2n - 2$.

Theorem 2 *(Bott periodicity)*

$$\Omega_a U \cong \Omega U \xrightarrow{\sim} \mathbb{Z} \times BU$$

This is because of the homology isomorphism and simple connectivity. Next time I'll talk about representations, so the reference will be Kac's group.

[Can one get real Bott periodicity?]

Yes, by working it out for O_n.

[Why doesn't the multiindex have to be nondecreasing?]

If I don't order them, I take a list of n of them. I could have ordered them, then $d(\underline{a})$ has a different formula.

Let's go for some coffee.

2 Teleman, twisted K-theory

So right now, I'm reviewing my notes this morning, one option was to cover everything I wanted and speed up massively, the other was to cross out some stuff, so I won't discuss twisted Chern characters.

So we duscussed models for BU. Recall $\lim\ Gr(n, 2n)$. with $Gr(n, 2n) \xrightarrow{\beta} \Omega_n U(2n)$.

Theorem 3 *Bott (Morse theory)*
β induces an isomorphism on homology and homotopy through a range increasing with n.

What I did not mention was a general theory, in the abstract definition of BU, where $U \cong \omega BU$ For any topological group G you have $G \to \Omega BG$ which is a map $SG \to BG$. Define a toutological bundle on $S^1 \wedge G$ by self-gluing the trivial bundle using the map $Id : G \to G$. Why does that give a homotopy equivalence?

$$\begin{tikzcd}
G \arrow[dr] & & \Omega(BG) \arrow[dr] & \\
& EG \arrow[rr, dotted] \arrow[d] & & P_{\cdot}(BG) \arrow[d] \\
& BG \arrow[rr] & & BG
\end{tikzcd}$$

So the space alternates between $\mathbb{Z} \times BU$ and BU. So we can define a 2-periodic spectrum with these spaces. We get $K^0(X) = [X, \mathbb{Z} \times BU]$, $K^1(X) = [X, U]$. The group law can be taken to be multiplication on U.

Proposition 1 *If X is compact, $[X, BU]$ the set of "stable isomorphism classees" of virtual vector bundles (of rank 0), a "stable isomorphism"' $V \overset{s}{cong} W$ if and only if $V \oplus \mathbb{C}^n \cong W \oplus \mathbb{C}^n$. Virtual vector bundles, formal differences of such classes form a group under $\oplus$.*

Fact: for a fixed n if X is compact, $[X, Gr(n, \infty)] \leftrightarrow$ isomorphism classes of n-dimensional vector bundles.

Over Gr_1 you have the tautological bundle. $p \in Gr(n, \infty)$ is an n-plane, that's the fiber of the tautological bundle.

The fact follows from, on a compact space, every vuctor bundle is a direct summand af some large trivial bundle.

Theorem 4 *(Swan)*
For X compact Hausdorff, projective modules, finitely generated, over $C^0(X)$ are in bijection with finite rank vector bundles.

Bundles map to the module of continuous sections.

Note that if V is a subbundle of $\mathbb{C}^n$ you get a map $X \to Gr(n, N)$ by $x \to V_x \subset \mathbb{C}^N$ and the pullback of the tautological bundle is V.

To increase the rank, you add a trivial line to your plane and the space. At that point you classify them up to stable isomorphism, not isomorphism.

Note that BU does not carry a finite dimensional tautological bundle, but it does carry a "virtual bundle" which is the tautological bundle, the limit of n-dimensional bundles, minus a trivial bundle of the same rank, ($\mathbb{C}^n$ on $Gr(n, 2n)$).

So or $Gr(n, 2n)$ the tautological bundle is naturally $V^{(n)} \oplus \text{trivial} - \mathbb{C}^n \oplus \text{trivial}$, and every map from compact X to BU will land in some finite $Gr(n, 2n)$ and we can pull back a virtual vector bundle.

Something with this theorem mest go wrong if X is not compact. From what I rememmber you can add the identity which was the one-point compactification.

To proceed we should mention the Atiyah index map

$$\Omega_{pol}U(N) \to GCr(N\infty, 2N\infty).$$

On the left are the loops with finite Fourier expansion, and on the right, well, the $2N\infty$ is the span of $z^k\mathbb{C}^N$) for $k \in \mathbb{Z}$ [unintelligible]is the subspaces sandwiched between $z^{-p}H_-$ and $z^q H_-$ for some p, q where H_- is the span for $z \leq 0$.

The map takes a loop to the image of H_-, the span of images of Fourier [unintelligible].

Note $\alpha \circ [\mathit{unintelligible}]$ embeds $Gr(n, n)$ as Gr of n-dimensional subspaces between H_- and zH_-, $N = 2n$.

Fact, $\Omega_{pol} \sim \Omega_{cont.}$ (Garland, Raghunatan).

The meaning of the maps $BU \to \Omega U$ corresponds to $S^1 \times BU \to U$. $\mathbb{P}^1 \times BU \to BU$ is the gluing map for a bundle on $\mathbb{P}^1 \times BU$.

Over $\Omega_{pol} \times \mathbb{P}^1$ have a universal holomorphic bundle (Glue along equator using loop in ΩU, δ-operator along $\mathbb{P}^1$ leads to an index bundle over [unintelligible]clrassified by α.

The meaning of [*unintelligible*] is the Thom isomoprhism from [*unintelligible*] $\times BU \to \mathbb{P}^1 \times BU$ and α is "integration along $\mathbb{P}^1$" in K theory. Ande $\alpha \circ$ [*unintelligible*] $= Id$.

To get things to work we change the model slightly, replacing $\mathbb{C}^\infty$ with a Hilbert space. Recall the norm-closure of finite rank operators on a Hilbert space are the compact operators, and the Gredholm operators are "invertible up to compact operators" with finite dimensional kernel and cokernel. They are bounded such that these facts are the same.

$Gr_{res}(H \oplus H)$ are subspaces such that the first projection is Fredholm and the second projection is compact. It's the closure of $\lim Gr(n, 2n)$. The space of matrices $GL_{res} = \left[\begin{array}{c|c} \text{Fredholm} & \text{Compact} \\ \hline \text{Compact} & \text{Fredholm} \end{array}\right] : H \to H$, bounded, invertible, we have $U(H)$ the unitary group, $U_k = U(H) \wedge (Id + \text{compact})$. A theeorem is $U(H)$ is contractible, $U_k \hookleftarrow U(\infty)$ is a homotopy equivalence.

Fact. We have an Atiyah map $\Omega_{pol} U_k \to Gr_{res}$ as before.

Fact. GL_{res} is homotopy equivalent to Gr_{res} thus $BU \times \mathbb{Z}$. And $GL_{res} \to Fred$ by the top left corner is an equivalence. So s[*unintelligible*]$\mathbb{Z} \times BU \cong GL_{res} \cong Gr_{res} \cong Fred$ which takes the components detected by the dimension of the kernel minus that of the cokernel ("index bundle") to the tautological bundle.

Proposition 2 *$\mathbb{P}UH = U(H)/U(1)$ is equivalent to $K(\mathbb{Z}, 2)$, that is $\mathbb{CP}^\infty$.*

The proof is contractibility of $U(H)$ so $U(H) = EU(1)$.

Remark, the multiplications are homotopy equivalent because $\mathbb{P}U(H) \sim \Omega(B\mathbb{P}UH) \sim \Omega K(\mathbb{Z}, 3) \sim K(\mathbb{Z}, 2)$ as a group. In the second place, π_* is $\mathbb{Z}$ in degree three so this is in $K(\mathbb{Z}, 3)$.

Observe $\mathbb{P}U$ acts everywhere an $Fred(H)$ $\mathbb{P}U(H)$ acts by conjugation, on $Gr_{res}(H \oplus H) \mathbb{P}U(H \oplus H)$ acts on the left. On $U_k(H)$ by conjugation. $\mathbb{P}U(H) \overset{diag}{\hookrightarrow} \mathbb{P}(U(H) \times U(H)) \hookrightarrow \mathbb{P}U(H \oplus H)$.

Theorem 5 *Every projective Hilbert bundle $\mathbb{P}H$ over X defines a twisted version of $K^*(X)$ by ${}^\tau K^* - \pi_* \Gamma(X, Fred(H))$. A class in K^0 is a Fredholm endomorphism of $\mathbb{P}H$ up to homotopy. Replace H with $H \otimes \ell^2$ if H is finite dimensional.*

Proposition 3 *projective Hilbert bundles up to isomorphism are classified by $H^3(X, \mathbb{Z}) = [X, K(\mathbb{Z}, 3)]$.*

Proposition 4 *If ${}^\tau K^0$ contains a section of virtual dimension k then k (the characteristic class of the bundle) is zero.*

So the last thing that I'd like to do is to explain the operations and the morphisms. How do you define addition, product, what structures are there? So I'll take a moment to explain that.

So actually, I made good progress here, look at that, actually no, I'd like to say the thing that will connect to what people will read in the literature.

The moral description of the twisting. $\mathbb{CP}^\infty \times BU \to BU$. think this is lines and vector spaces to vector spaces, so the multiplication is a tensor. Formally this map is defined by a bundle over $\mathbb{CP}^\infty \times BU$. Which one, This is $\mathbb{P}U \times Fred \to Fred$. So $[U]F \to UFU^{-1}$. So there's a phase ambiguity det $\boxtimes$index bundle $\to$ index bundle is det over $\mathbb{P}U$ in correspondence with $U(H) \hookrightarrow$ over $\mathbb{P}U$.

Heuristic model for twisting and K-theory is that a $\mathbb{P}U$ bundle is a "one-cocycle ω with values in lines." So if $\sqcup V_j \to X$ is a cover we want $L_{ij} \to V_i \cap V_j$. The cocycle condition means L_{ii} is $\mathbb{C}$ and $L_{ij} = L_{ji}^{-1}$ with $L_{ij} \otimes L_{jk} \otimes L_{ki} \sim \mathbb{C}$.

On a quadruple intersection

$$\begin{tikzcd} L_{ij} \otimes L_{jk} \otimes L_{k\ell} \arrow[r] \arrow[d] & L_{ij} \otimes L_{j\ell} \arrow[d] \\ L_{ik} \otimes L_{k\ell} & L_{i\ell} \end{tikzcd}$$

We want this to commute. A twisted class is a virtual vector bundle E_i in each V_i with isomorphism $L_{ij} \otimes V_j \leftarrow V_i$ on the overlap.

3 McClure, Operads

Let me remind you what was the end of the first lecture. We had

Theorem 6 *Y has a grouplike action of an A_∞ non-Σ operad if and only if Y is weakly equivalent to ΩZ for some Z.*

This was proven in a particular case by Stasheff. Then with connected Y this was Boardman-Vogt, and finally in full generality with a different proof by May.

How do you find Z? If Y has a strictly associative multiplication, define BY to be the geometric realization of the following space: $Y \times Y \times Y$
$Y \times Y$
Y

So this is a special case, so, in this case with a simplicial object, here d_0 is projection, the last one d_{last} also projects away the last coordinate, and d_i multiplies the i and $i+1$ entries. The other maps introduce degeneracies.

Then $Y \cong \Omega BZ$ if Y is group complete.

I think this simplicial object is the single most significant thing about simplicial sets, this bar construction. The simplicial relations will only be satisfied up to homotopy if Y is not strictly associative.

In the general case you can do it, but it's harder. You can think of this as an A_∞ simplicial object and then use a Segal pushdown to make it actually simplicial. There are many other ways of doing this.

I want to talk now about the Stasheff operad, $\mathscr{K}$, it's general features and what makes it interesting.

It's small and explicit. So $\mathscr{K}(k)$ is a polyhedron, which is nice because it's finite dimensional. Further BY has a simple description, you take $(\sqcup\mathscr{K}(k) \times Y^k)/\sim$, and see Stasheff's paper for more details.

There's also an obstruction theory. There's a step by step process to create an A_∞-algebra action on Y which involves only extending an action on the boundary to the interior. There was a bunch of work on this showing some things about spectra.

The obstructions in this case are Hochschild homology groups, so you can calculate them.

So far I haven't said anything about A_∞-algebras, and I thought it would be a nice gesture to say something. For this I need to talk about, well non-Σ operads in the category of chain complexes Ch. The most common category to use for an operad other than spaces is chain complexes. Replace the cartesian product with the tensor product in the category of chain complexes. I'll say a little about it. We want to have, we want to have, oh, you know, I was talking about the advantages of the Stasheff operad, we use other things because sometimes it's nice to have things that are big and functorial.

So a chain operad will be $\mathscr{P}$ and I want a map $\mathscr{P}(k)\otimes\mathscr{P}(j_1)\otimes\cdots\otimes\mathscr{P}(j_k) \to \mathscr{P}(j_1+\cdots+j_k)$ satisfying the same things.

An A_∞ chain non-Σ operad is one in which each chain complex $\mathscr{P}(k)$ has the homology of a point. Then an A_∞ algebra (that was my official topic, now they'll reimburse my expenses), is a chain complex with an action of a chain A_∞ non-Σ operad. Analogously you can say an A_∞ space is a space with an action of an A_∞ non-Σ algebra. This is good if you come upon things that are not strictly associative in nature. You can get Ext and Tor and so on, you have a bar construction.

Remark: the Stasheff operad is a cellular operad, it has a cellular decomposition compatible with the operad structure, this is another reason the Stasheff operad is improtant. Tde cellular chains of $\mathscr{K}$ are a small A_∞ chain non-Σ operad.

All right, having said this much, let me point out that we kind of don't need A_∞ algebras at all. This is the subject of rectification. There's a functor taking spaces or chain complexes or spectra or any model category with an A_∞ action to weakly equivalent spaces and so on with a strictly associative multiplication. So we can take this A_∞ multiplication with higher stuff going on to a strictly associative one.

This shouldn't be surprising, you've heard of the Moore loop space? The Moore loop space of Z is strictly associative and homotopy equivalent to ΩZ. So why do we bother with A_∞ things at all? Normally you get things from the context with information and questions. That destroys a bunch of information. The rectification result is useful heuristically. It means anything you can do for a strictly associative thing can be done for A_∞ things as well.

Another name for operads would be non-non-Σ operads, the motivation would be that non-Σ operads encode higher associativity information, but we'd also like to encode higher commutativity. A very simple example is the two-loop space. π_0 is commutative because it's π_2 of the original space.

From the elementary point of view, a k-fold multiplication is commutative if you can permute its inputs and get the same answer. For higher commutativity we want it to be homotopic in some nice way. Before I give the definition, I'll give an example. Let Y be any space. Let $\mathscr{O}(k) = \mathrm{Maps}(Y^k, Y)$ with the following structure

1. $id \in \mathscr{O}(1)$
2. composition operations
3. Σ_k action of $\mathscr{O}(k)$ (induced by the action on Y^k)

Exercise 1 *Find two relationships between the Σ_k action and composition. These are in May's book.*

This is called the endomorphism operad of Y, it plays an important role.

Definition 4 *An operad $\mathscr{O}$ is a non-Σ oerad together with, for each k, an action of Σ_k on $\mathscr{O}(k)$ satisfying those two relations that I didn't write down*

Definition 5 *Let $\mathscr{O}$ be an operad and Y a space. An action of $\mathscr{O}$ on Y is an action of the underlying non-Σ operad such that each map*

$$\begin{tikzcd} \mathscr{O}(k) \times Y^k \arrow[rr] \arrow[dr] & & Y \\ & \mathscr{O}(k) \times Y^k/\Sigma_k \arrow[ur] & \end{tikzcd}$$

So it factors through.

Exercise 2 *An action of $\mathscr{O}$ on Y as the same as an operad morphism of $\mathscr{O}$ to the endomorphism operad of Y.*

It's natural to ask what happens when all the spaces are contractible.

$\mathscr{O}$ is E_∞ if each $\mathscr{O}(k)$ is weakly equivalent to a point. E stands for everything, associativity and commutativity, and this terminology is due to Boardman-Vogt.

Theorem 7 *(Boardman-Vogt, May)*
Y has a groupike action of an E_∞ operad if and only if Y is an infinite loop space. That is there exist $Y_1, Y_2, \ldots$, with $Y \overset{\text{wk}}{\cong} \Omega Y_1, Y_1 \overset{\text{wk}}{\cong} Y_2, \ldots$

This gets beginners confused: you cannot replace spaces with an E_∞ action by weakly equivalent strictly commutative spaces. There are lots of infinite loop spaces like BU which are not products of Eilenberg-MacLane spaces. So there's no rectification.

[The people back here want you to call that a theorem.]

Call it whatever you want in your notes, I'm not going to be checking.

Over the rationals, you can rectify commutativity, rational homotopy is taking the E_∞ algebra of cochains and replacing it with a differential graded algebra.

An example of an E_∞ operad is the linear isometries operad (Boardman-Vogt). There's actually a linear isometries PROP. It was Peter May who crystallized that operads were important as a subidea of PROPs. You can't write down the free algebra over a PROP like you can over the algebra.

Let $\mathbb{R}^\infty = \cup \mathbb{R}^n$, and then this has an inner product. Let $\mathscr{I}(k)$ be spaces of linear maps $(\mathbb{R}^\infty)^{\oplus k} \to \mathbb{R}^\infty$ respecting the inner product. Operad composition is ordinary composition as a subset of the pieces of the endomorphism operad of $\mathbb{R}^\infty$.

Exercise 3 *$\mathscr{I}(k)$ is contractible, so $\mathscr{I}$ is an E_∞ operad.*

You might have noticed that I've given theorems about operad actions giving a structure, I haven't used those. Let $O = \cup_{n \geq 1} O(n)$, where the inclusions are by putting a 1 in the corner with 0 in the other new places. We have BO which you can get as I did before.

Exercise 4 *$\mathscr{I}$ acts on BO.*

So BO is an infinite loop space. We know this, but we can replace $O(n)$ with $Top(n)$, and then that's an infinite loop space and you get something that you can't get this from Bott.

There are lots of things you can do like this.

In the remaining fifteen minutes I'd like to talk about the little n-cubes, because I think that fits with a lot of other things that have been talked about this week.

I'll use 2 for specificity.

In the little intervals operad, an element was a collection of intervals with nonoverlapping interiors. A point in the little intervals operad $\mathscr{C}_2(k)$ is a collection of k closed squares in the unit square with sides parallel to the coordinate axes with nonoverlapping interiors which are numbered (any way you like). There was a natural numbering for the little intervals operad. If I had allowed a symmetric action that would have made it an operad, not non-Σ. This space is not contractible but its homotopy type is very well understood. Lots is known about this space.

To describe the composition, it's like, you can, I'll give a representation of it. Now $\mathscr{C}_2$ is supposed to act on $\Omega^2 Z$. For $\mathscr{C}_n$ you just take n-cubes. This should act on $\Omega^2 Z$ as follows, this is all kind of also an exercise. Look at the element I drew on the board in $\mathscr{C}_2(4)$. call it x. Now x gives a map from $(\Omega^2 Z)^4 \to \Omega^2 Z$ by $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, well, I think of an element of $\Omega^2(Z)$ as a map $(I \times I)/\partial \to Z$. I just put my αs into the squares in the picture, and I take everything else to the basepoint. That sits inside of $\mathrm{Maps}((\Omega^2 Z)^k, \Omega^2 Z)$.

Exercise 5 *This family $\mathscr{C}_2(k)$ is closed under multivariable componsition, describe the composition exactly.*

Theorem 8 *(Boardman-Vogt, May)*
Y has a grouplike action of the little 2-cubes operad if and only if Y is weakly equivalent $\Omega^2 Z$ for some Z.

This is true for n as well. But there is an open research problem, how do you tell if another operad is weakly equivalent to $\mathscr{C}_n$? that's not known for 3 and above.

For A to be equivalent to B means that you have maps in a finite chain inducing isomorphisms on homology, in operads on each space $\mathscr{O}(k)$.

In the last, I guess I'm out of time, I'll stop.

Ask me in the discussion session how to find a Gerstenhaber structure on a twofold loop space.

[How do you find Z?]

The only way known is a huge monadic bar construction. That's related to the problem of iterating the cobar construction.

4 Discussion

The question, recall, the topolog on $L_a K$ was $\varinjlim L_a^m K$ where $L_a^m K = \{A(z) = \sum_{i=1}^m A_i z^i\}$ for $A_i \in End(\mathbb{C}^n)$.

If $K \subset U(n')$ then consider

$$\begin{tikzcd} K \arrow[r, "\Delta"] & U(n) \times U(n') \arrow[d] \arrow[dr] & \\ & U(n) & U(n') \end{tikzcd}$$

The direct limits will be the same.

[Can you tell us more why we are interested in loop groups?]

The representation theory is close to that of semisimple groups. There are a lot of things that you can do that yield many interesting formulas. Before Kac were the MacDonald identities, combinatorial identities, which were special cases of the Kac character formula. There is the whole physical aspect: the right kind of CFT with the right kind of symmetry, the loop group and even the Virasora algebra will act on it.

[Why are topologists interested?]

The equivariant elliptic cohomology of a point with respect to a compact Lie group is closely related to LG.

[Do loop groups have anything to do with 2-categories?]

I'm sure they do. There is a recent paper by Stevenson.

[Is a justification for why these are related based on K-theory?]

There's something where he calculates the rational equivariant cohomology. There's a model for this cohomology, you just formally manipulate it, you can work with maximal tori and the affine Weyl group, and rationally it's canonically isomorphic with the characters of irreducible representations of the loop group.

[Something about pairs of commuting elements of the group or loop group]

According to Hopkins, [unintelligible], BG are related to these things, this should have to do with field theories as maps of tori into BG.

This is due to Witten, [unintelligible]holomorphic sections of line bundles, [unintelligible]commuting holonomies, I think what was missing then, the level has something to do with it, it is a kind of twisting, the different levels, that's the new insight.

If you don't twist you get nothing at level zero. Witten '88 or something, hidden in there.

Matthew Ander, preprint, is the reference. I think it's on his webpage

The other thing is if you localize, you get [unintelligible], and that's the right answer.

4.1 Teleman

[I want to ask, can you give us [unintelligible]?]

I want to say one thing before that, you have Mayer Vietoris for ${}^nK(S^3)$ so $H^3 = \mathbb{Z} \ni n$ so $0 \to^n K^0(S^3) \to^n K^0(D^+) \oplus^n K^0(D^-) \to^n K^0(S^2) \to^n K^1(S^3) \to 0$.

The disks we know we get $\mathbb{Z}^2$ and then $\mathbb{Z} \oplus \mathbb{Z}^{L-1}$ where the map to it is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. [unintelligible]if you trivialize over the two hemispheres, they are glued by a nontrivial map $\mathbb{P}U \to \mathbb{P}U$. The gluing $S^3 \to \mathbb{P}^2$ has degroo n. The trivialization on S^2 is by D^+ and the one from D^- differs by the action above. So what is the effect of the change of the $\mathbb{P}U$-bundle on K-theory? Well $[X, \mathbb{P}U] = H^2(X, \mathbb{Z})$ which are isomorphism classes of line bundles. So we guess the change af trivialization is in correspondence with tensoring by the corresponding index bundle.

So the new map ${}^nK^0(D^-) \to K^0(S^2)$ is the old map $\otimes L^{\otimes n}$. This is $(1 + (L-1))^{\otimes n} = 1 + n(L-1)$. So this is $\mathbb{Z}^2 \overset{\begin{bmatrix} 1 & 1 \\ 0 & n \end{bmatrix}}{\to} \mathbb{Z}^2$.

This is now injecive with quotient $\cong \mathbb{Z}/n$ so trivialize on S^2 as in D^+ and ${}^nK^0(S^3) = 0,^n K^1(S^3) = \mathbb{Z}/n$.

Look at a real vector bundle V with $W_3 \neq 0$. So $S(V)$ is a projective bundle. Then $S^+ \to S^-$ tells you that the Thom class does exist in twisted K theory ${}^{W_3}K(DV, \delta DV)$.

If the dimension of V is 3 then $1 \to U(1) \to U(2) \to SO(3) = PU(2) \to 1$. So the spin bundle is then naturally $\mathbb{C}^2$. You can't construct it as a bundle because of phase ambiguity, but you can do [unintelligible]projectively. This leads to a $\mathbb{P}^1$-bundle used to define twisting.

[Give something nontrivial?]

The tensor product, behind it something will be nontrivial. You have the tensor product inducing the ring structure on K-theory $K^0(X) \times K^0(X) \to K^0(X)$. With twisting you get $\mathbb{P}(H_1), \mathbb{P}(H_2) \to$ projective Hilbert bundles. We can then define $\mathbb{P}(H_1 \otimes H_2)$.

So now if F_1, F_2 are Fredholm endomorphism representation classes in ${}^{\mathbb{P}_1\mathbb{P}_2}K^0$ then

$$H_1 \otimes H_2 \overset{F_1 \otimes 1, 1 \otimes F_2}{\to} H_1 \otimes H_2 \oplus H_1 \otimes H_2 \overset{1 \otimes F_2, F_1 \otimes 1}{\to} H_1 \otimes H_2$$

. This defines the multiplication ${}^{\mathbb{P}_1}K \otimes^{\mathbb{P}_2} K \to^{\mathbb{P}_1 + \mathbb{P}_2} K$.

Observe that the class of $\mathbb{P}(H_1 \otimes H_2)$ is the sum in $H^3(X)$ of those of the factors.

Suppose that ther was a twisting by $\{\pm 1\}$ implemented by the Thom twists of line bundles. $Y^n \to \underline{Omega}Y^{n+1}$ (this is the $\mathbb{Z}/2$ action. Then we get twistings classified by $H^1(X, \mathbb{Z}/2)$. Let's do it correctly.

Addition of Thom twistings

$$\begin{tikzcd} R_1 \arrow[dr] & & R_2 \arrow[dl] \\ & X & \end{tikzcd}$$

real line bundles. To define the Thom twist for the total bundle $Y^n \xrightarrow{\sim} \underline{\Omega\Omega} Y^{n+2}$ where these are $\mathbb{Z}/2$ actions. We get a bundle of spectra equivalent to the addition of the Thom twists. Now naive addition gives ΩY^{n+1} coupled to $R_1 \otimes_{\mathbb{R}} R_2$ which gives the wrong answer. $R_1 \oplus R_2 \nrightarrow R_1 \otimes R_2 \oplus \mathbb{R}$ because W_3 of the first part can be nonzero.

So $W_3(R_1 \oplus R_2) = \delta(W_1(R_1)W_1(R_2)$ where $\delta : H^2(X, mathbbZ/2) \to H^3(X, \mathbb{Z})$.

Thbe twistings that we have are classified by $H^1(X, \mathbb{Z}/2)) \times H^3(X, \mathbb{Z})$ but addition is $(a_1, a_3) + (b_1, b_3) = (a_1 + b_1, a_3 + b_3 + \delta(a, b))$.

[You have not mentioned the twisted cohomology where [unintelligible]lives.]

There's not a very easy [unintelligible], you have to work rationally. $ch : K(X) \otimes \mathbb{Q} \to H^0(X) \otimes \mathbb{Q}$ which is a ring isomorphism if X is a finite complex.

Here you get ${}^{\tau}ch :^{\tau} K(X) \to^{\tau} H^*(X) \otimes \mathbb{R}$ (graded mod 2). If X is a manifold, then this is $(\Omega^{\cdot}(X), h + \eta\wedge)$, where this is the differential. [unintelligible][unintelligible]trace of operators. I don't know what kind you need, but it's a doable process, [unintelligible]rational homotopy theory, Sullivan's minimal models.

[This cocycle is only defined up to cohomology.]

Up to noncanonical isomorphism, so it depends on if you care about isomorphism, for example if you are in the Mayer Vietoris.

[What about the Thom isomorphism?]

It exists. Do the original construction, the mechanics are the same, but remember they are projective now and then add the twisting.

You can use it to diefine $f! :^{\tau'} K^*(X) \to^{\tau} K^*(Y)$ between compact manifolds if you have been given an isomorphism between τ' plus the Thom twist of $TY - TX$ and $f^*\tau$.

[This group law on $H^1 \otimes H^3$ must correspond somehow to $\mathbb{RP}^{\infty}$.] The piece of twisting will fiber over $K(\mathbb{Z}/2, 1)$ with fiber $K(\mathbb{Z}, 3)$. It comes from the bottom of BO I think. It comes from the class of δ, it's a pushout of an extension. That's not a K-invariant. There is one at $K(\mathbb{Z}, 4)$ fibering over $K(\mathbb{Z}/2, 2)$. In H^5 of this guy. You expect to have this kind of extension.

Let's thank Constantin.

4.2 McClure

Can you give me a summary of all the operads, which ones are equivalent, et cetera.

For non-Σ A_∞ operads, all of these are equivalent, you have the unlabelled little intervals operad, the Stasheff operad. Of course any E_∞ operad is one forgetting the Σ-action, such as the linear isometries operad.

There's one that comes from trees, I may not get this quite right, $\mathscr{O}(k)$ is the nerve of the category of binary trees (two branches at each vertex) with k leaves and exactly one morphism between each pair in each direction. The interesting thing about this one, I remember really being struck, is it's related to MacLane's thing for monoidal categories. So it acts on the realization of $|N\mathscr{A}|$ where this category is monoidal, this is MacLane's coherence theorem. I was struck by that, there's a similar thing for the E_∞.

We also have a cosimplicial one.

For E_2, things that are weakly equivalent to the little 2-cubes, you have C_2, the cactus operad (which was like the framed one, so let's call it the unframed one). Jeff and I do have a model which is useful, McClure-Smith, isomorphic to the Kauffman model. He started from the arc complex for the moduli space of Riemann surfaces, ours was related to the Hochschild [unintelligible].

For C_n I don't know any for $n > 2$. We have a model, McClure-Smith, this is all in the survey paper.

For E_∞ there's a ton. Linear isometries, there's an analog of the tree model for A_∞, which corresponds to the symmetric monoidal coherence theorem. This is at the heart of Segal's paper on [unintelligible].

There's the Barratt-Eccles, where the kth space is $E\Sigma_k$. Jeff Smith's thesis has a model which is different. Another one is $\mathscr{C}_\infty$, the colimit of the $\mathscr{C}_n$, which is too horrible geometrically to contemplate, then McClure-Smith, have I written my name down enough.

Then there's the framed case where you have framed versions of all of these things.

The Lie_∞ operad also exist, not on a space level. That's an important example. A Lie_∞ algebra was inspired by mathematical physics.

[The thing about Gerstenhaber.]

Theorem 9 *If Y has an action of C_2 then H_*Y is a Gerstenhaber algebra.*

This is due to Cohen, parts are due to other people, Fred Cohen in his thesis. He created a huge structure in his thesis. We start from the map $\mathscr{C}_2(2) \times Y \times Y \to Y$. Now $C_n(2) \cong S^{n-1}$ so this is S^1 (Do this as an exercise. Shrink to get just two points. The k-th space is homotopy equivalent to a certain configuration space. This gives a vector which I normalize to get an element of S^1. That's a homotopy equivalence. In homology, what was I calling the action?

I have the map ϕ, so in homology I get $\phi_* : H_*(S^1) \otimes H_*Y \otimes H_*Y \to H_*Y$. Now $H_*(S^1)$ has two generators ι_0, ι_1. Now $\phi(i_0 \otimes y_1 \otimes y_2)$ is $y_1 \cup y_2$ which is a disguised Pontryagin product.

$\phi(i_1 \otimes y_1 \otimes y_2)$ is $[y_1, y_2]$. So the Jacobi identity is hard, and the Poisson compatibility.

[Say a word about the BV?] For that you need the framed little disks. Now I have disks instead of cubes. I can use disks instead of cubes to define $\mathscr{C}_2$ if I want to. That's homotopy equivalent, take the inscribed square or whatever. Getzler pointed out that here you can rotate. So you can do a semidirect product with S^1. So you put a rotation coordinate on each one. I won't define the composition, it's a little tricky but not hard.

So framed little disks is little disks, each labelled with an angle, let's say. I don't know a standard name, we can call it $\mathscr{F}_2$. now $\mathscr{C}_2 \subset \mathscr{F}_2$. There is also a projection, which is not an operad map.

Theorem 10 *Getzler*
If Y has an $\mathscr{F}_2$ action then H_*Y is a BV algebra

This is $\cup, [\ ], \Delta$. The hard part is the relations, So Δ, well $\mathscr{F}_2(1)$ is homotopy equivalent to S^1 by moving the circle to the center. Then I have $\mathscr{F}_2(1) \times Y \to Y$. So $H_*\mathscr{F}_2(1) \otimes H_*Y \to H_*Y$. So ι_0 is the identity here and ι_1 gives the Δ.

[Can you characterize spaces with $\mathscr{F}_2$ actions?]

That's a good question, I think [unintelligible]did.

[If you have an action of the Stasheff operad you can get Z back, with some kind of bar construction. Does that mean two operads can be equivalent but their bars are not?]

It just makes it easier to write down. Within one homotopy class you can have a bunch of algebraic or geometric complexity, so that's what happens. Trying to find Z is an interesting property in a number of settings.

[You said something about A_∞ simplicial sets. What's a reference?]

The appendix in Segal's original paper on Categories [unintelligible].

[Additional questions? Let's thank the speaker.