# Stringy Topology Notes
# January 11, 2006

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Good morning, I have figured out the thing about the excursion for Saturday, there is information on the board, it will be 150 Pesos. Please get your ticket. We only have forty places. I already discussed where we are going yesterday.

We start today with Teleman, teaching us about twisted $K$-theory.

## 1 Teleman, Twisted $K$-theory

I would like to thank the organizers for inviting me to this, my first time in Mexico and I like it already.

I want to talk about twisted $K$-theory, it's easy to give the model, the axioms. There are a lot of different variations, and the translations between them are a little difficult. Today I thus want to do it a little more abstractly. Tomorrow I'll talk about the twisted Chern character and maybe equivariant $K$-theory. There will be some overlap, I hope it doesn't work modulo 2, that is, cancel out previously acquired knowledge.

Recall cohomology $H^*$ is

- a functor from compact (Hausdorff) topological space pairs $(X, A)$ to Abelian groups $H^k(X, A; \mathbb{Z})$
- it is homotopy invariant, $f \sim g$ from $(X, A) \to (Y, B)$ implies $f^* = g^*$
- "Excision" $H^*(X, A; \mathbb{Z}) \cong H^*(X/A, A/A; \mathbb{Z}) = \tilde{H}^*(X/A, \mathbb{Z})$ "missing a $\mathbb{Z}$ in $H^0$ from the regular cohomology.
- Mayer-Vietoris, $X = U \cup V, U, V$ closed then we have a long exact sequence

$$\longrightarrow H^*(X) \longrightarrow H^*(U) \oplus H^*(V) \longrightarrow H^*(U \cap V) \longrightarrow$$

- The cohomology groups of a point are a single $\mathbb{Z}$ in degree zero.

**Theorem 1** *(Eilenberg-Steenrod) These properties determine $H^*$ for finite cell complexes.*

This implied that the many constructions of cohomology gave the same answer for such spaces.

Fifty years ago they were studying the fine differences. Eventually they started losing some axioms. The interesting one to lose is the dimension axiom, the cohomology of a point. If the answer is $M$ in degree zero then this gives $H^*(X; M)$, the cohomology with coefficients in the Abelian group $M$.

## 1.1 Generalized cohomology theory

These satisfy all axioms except the "dimension axiom." It's easy to produce the "negative half" of such a theory. For "any" space (really we want cell complexes in the end) $Y$ with base point $y$ we define $h^k_Y(X) = \pi_{-k}(\mathrm{Maps}(X, Y))$ and $h^k_Y(X, A) = \pi_{-k}(\mathrm{Maps}(X/A, A/A; Y, y))$ for $k = 0$.

Recall $\pi_n(Z) = \mathrm{Maps}(S^n, n; Z, z)/\text{homotopy}$, called $[S^n, Z]$.

Recall that for $X$ with base point $x$, $S^n \wedge X = S^n \times X/S^n \times \{x\} \cup n \times X$. Also $\Omega^n X = \mathrm{Maps}(S^n, X)$ which send $n$ to $x$.

**Proposition 1** $\pi_{-k}(\mathrm{Maps}(X/A, Y)) = [S^{-k} \wedge X, Y] = [X/A, \Omega^{-k} Y], \ k \leq 0$

So all definitions describe maps $S^{-k} \times (X/A) \to Y$. You just have to check that the restrictions match.

We know that $h^0_Y$ is a set, $h^{-1}_Y$ is a group, and $h^{\leq -2}_Y$ is an Abelian group. Further, all the axioms hold where they can, including Mayer Vietoris. Let me write the connecting map for Mayer-Vietoris. This is to go $h^{-k}_Y(U \cap V) \to h^{1-k}_Y(U \cup V)$. So we want a map $[S^k \wedge (U \cap V), Y] \to [S^{k-1} \wedge (U \cup V), Y]$. This will be independent of $Y$. Let $X'$ be copies of $U$ and $V$ connected by $(U \cap V) \times I$. This maps to $S^1 \wedge U \cap V$ by smashing the $U$ and $V$ to a point. But if $U, V$ are "reasonable" (e.g. cell subcomplexes) then $X'$ is homotopy equivalent to $X$. Then $S^{k-1} \wedge X \sim S^{k-1} \wedge X' \to S^{k-1} \wedge (S^1 \wedge (U \cap V)) = S^k \wedge (U \cap V)$. This induces the connecting map between homotopy classes of maps to $Y$.

How do you continue to positive degrees? In general you can't. If $Y \sim \Omega Y^1$, that is, based maps $S^1 \to Y^1$, then $h^{-k}_Y(X) = [X, \Omega^k Y] = [X, \Omega^k \Omega Y^1] = [X, \Omega^{k+1} Y^1]$.

Now you can define it for $k = -1$ as $[X, Y']$ so $h'$ is defined. Now $h^0$ is a group, $h^{-1}$ is Abelian, and so on. With machinery you can see that this can only be extended if $Y$ can be written as a loop space.

**Theorem 2** *Full generalized cohomology theories are in bijection with "infinite loop spaces" (related to spectra) $Y \sim \Omega Y^1 \sim \Omega^2\ Y^2 \sim \Omega^3 Y^3 \sim \cdots$. You must specify the sequence of spaces and the maps (homotopy equivalences) to define $h^k_Y$ completely in all degrees.*

For ordinary cohomology these are Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$. Exceptionally they are uniquely defined up to homotopy equivalence by the condition $\pi_n(K(\mathbb{Z}, n)) = \mathbb{Z}$ and there are no other homotopy groups. This follows from the dimension axiom, if you [unintelligible]

Clearly $K(\mathbb{Z}, n) = \Omega K(\mathbb{Z}, n+1)$ so it's a group, homotopy commutative (easy) and harder, each is a topological Abelian group.

$K(\mathbb{Z}, 0) = \mathbb{Z}$.
$K(\mathbb{Z}, 1) = S^1$
$K(\mathbb{Z}, 2) = \mathbb{CP}^\infty = \mathbb{C}^\infty \backslash \{0\}/\mathbb{C}^* = S^\infty/U(1)$. Now $S^\infty$ is contractible so $\pi_*(\mathbb{CP}^\infty) \cong \pi_{*-1}(U(1))$.

Take for $\mathbb{C}^\infty$ the space of rational functions on $\mathbb{P}^1$. You can multiply them so they form an Abelian group. Divide by $\mathbb{C}^*$ and it is still okay.

Recall a rational function can be determined up to scale by 0s and poles with multiplicities. So this is "$SP^\infty \mathbb{CP}^1$." Let's not get into the cardinality of the dimension or the topology. This is configurations of points with integer labels. You require that most labels are zero.

**Theorem 3** *(Dold-Thom)*
$K(\mathbb{Z}, n) \cong SP^\infty(S^n)$.

Most interesting generalized cohomology theories do not come from topological Abelian groups.

**Theorem 4** *Every topological Abelian group "is" a product of Eilenberg MacLane spaces, perhaps not for $\mathbb{Z}$*

If homology theories are maps into a space, then twisted theories are twisted versions.

A twisted version of $Maps(X, Y)$ is $\Gamma(X; \underline{\underline{Y}})$ the space of continuous sections of a fiber bundle over X with fiber $Y$. So $p^{-1}(x) = Y$ at each $x$ and locally $p^{-1}(U) = U \times Y$ for small $U$. Call $\tau$ for twist.

Then ${}^\tau h^{-k}_Y = \pi_k(\Gamma(X; \underline{\underline{Y}})$, possibly based.

Remark: We will want a section of the bundle to act as the base point of the space of sections. This only defines the negative groups. To get the higher ones, you want every fiber to be a loop space, $\underline{\underline{Y}}$ should be the bundle of vertical loops in a bundle $\underline{\underline{Y}}^1$, and that be a bundle of vertical loops of $\underline{\underline{Y}}^2$ and so on.

Example: Say you have a topological group $G$ acting compatibly on the spaces $Y, Y^1$, etc. of the spectrum ($\Omega Y^1 \sim Y$ with $G$-action, etc.) and a principal $G$-bundle over $X$, called $P$. That's a fiber bundle with freely acting fiber $G$ with quotient $P/G = X$.

Now finally you get the structure. $\underline{\underline{Y}}^k = P \times \underline{\underline{Y}}^k / Diag(G)$. I should have said $G$ must fix all the base points. This is a typical way to find twisted cohomology theories. I will skip to get to $BU$.

Fact, $\{\pm 1\}$ always acts on cohomology theories, on spectra. You can get a twisted version associated to double covers $\tilde{X} \to X$. You get $H^*(X, \mathbb{Z}(\tau))$.

There is a version of Mayer Vietoris. I will discuss it tomorrow.

Main example. $BU$ and the "$\mathbb{CP}^\infty$ action"
$BU \times \mathbb{Z}$ is the zeroth space of the spectrum representing $K$-theory, topological, complex. But there are several models which are concrete representing this. So what is this?

1. $U = U(\infty) = \cup_{n\in\mathbb{N}} U(n)$ with $U(n) \hookrightarrow U(n+1) \hookrightarrow \cdots$
2. $BU$ is the base of the universal principal $U$-bundle $[X, BU]$ is isomorphism classes of principal $U$-bundles over $X$.
   Equivalently there exests $EU$ over $BU$ a contractible principal bundle over it.
3. $BU = \cup BU(n)$ where $BU(n)$ is to $U(n)$ what $BU$ is to $U$.
4. $BU(n) = Grass(n, \infty) = \lim_{m\to\infty} Gr(n, n+m)$. so $BU \overset{lim}{=\to} nGr(n, 2n)$.
5. $BU(n)$ classifies isomorphism classes of $n$-dimensional complex vector bundles and $[X, BU(n)]$ are isomorphism classes of rank $n$ vector bundles.

**Theorem 5** *(Bott periodicity)*
*$\mathbb{Z} \times BU \to \Omega U$ where $\pi_1(U) = \pi_1(U(1)) = \mathbb{Z}$ so $\pi_0(\Omega U) = \mathbb{Z}$.*

Let me write the map and stop. The Bott map $\beta : Gr(n, 2n) \to \Omega U(2n)$ takes $p$ to projection $P$of rank $n$ in $\mathbb{C}^{2n}$ to $\gamma_P(t) = exp(2\pi i t P)$

[Twisted cohomology is a functor from what to what?]

It's from pairs of spaces with a twisting to Abelian groups.

Homotopies must lift to a map of principal bundles. You must specify the lift.

# 2 Godin

The goal in string topology is to study the topology of the loop and path spaces of the manifold.

For this entire talk $M$ will be a smooth oriented manifold of dimension $d$. The spaces we're going to study, the main one is $LM = \text{Maps}(S^1, M)$. I don't fix a base point. One related subspace is $\text{Maps}_*(S^1, M)$, which are based. The goal of today is to study $H_*LM$. We'll define a product, a bracket, different operators, which will come from one idea, how to get intersection theory to work on this infinite dimensional space. I'll start informally and then be more formal and eventually get into details. This is mostly from "String Topology" by Chas and Sullivan. This will never be published. It's almost better that way, you can get it off the web.

First I'm going to give the idea behind the product, called the loop product. If you take a free loop, you can look at the value it takes at the basepoint of the circle $S^1 = I/\delta I$. You have

$$\begin{tikzcd} \Omega_x M \arrow[r] & LM \arrow[d, "ev_0"'] \\ & M \end{tikzcd}$$

- $\Omega M$ has a product, the concatenation of loops.
- $M$ has an intersection product $H_*M \otimes H_*M \to H_{*-d}M$.

The idea is to combine these to get the loop product.

Say you have two chains in the loop space $\alpha : \Delta^p \to LM, \beta : \Delta^q \to LM$. We get $ev_*\alpha : \Delta^p \to M$, $ev_*\beta : \Delta^q \to M$. If the basepoint chains intersect transversally, you can restrict to where they do so in $M$. Take $A \subset \Delta^p \times \Delta^q$ where $\alpha$ and $\beta$ start at the same point (under $ev_*$), then the loops are composable. So $\alpha \times \beta|_A \subset \text{Maps(figure eight}, M)$ which maps to $LM$. So you get $\alpha \bullet \beta : A \to LM$ which is a $p + q - d$ dimensional chain of $M$.

Out of this simple idea you get a full structure on the loop space.

This is the idea but it's not quite formal because you need some transversality arguments.

The next thing is just a brief review of intersection theory in $M$. One way to do it is with Poincaré duality but $LM$ is infinite dimensional so we don't want to do it that way. Now for an $\mathbb{R}^n$-bundle $\begin{array}{c}\mu \\ \downarrow \\ X\end{array}$ we take $D_\mu, S_\mu$ to be the disk and sphere bundles. Then $D_\mu/S_\mu$ is the Thom space. For example, if you take the trivial line bundle over $X$, the disk bundle is an interval with the sphere bundle two segments (sections). So the identification smashes the ends of the cylinder over $X$ to a point.

So let me talk about the Pontryagin Thom class. Start with $P \overset{e}{\hookrightarrow} M$. You have $\nu = TM|_P/TP$, the normal bundle of $e$. There is the tubular neighborhood theorem in five dimensions that says you have a neighborhood $U_P$ around $P$ identified with the normal bundle $D\nu$. You then take everything outside the neighborhood and collapse it to a point.

[Some contention about whether the neighborhood should be open or closed, all pedantic and semantics.]

By collapsing everything outside you get a map from $M$ to $M/M - U_P$. You collapse the boundary of the tube (disk bundle) and so get the Thom space. This is the perfect tool to look at intersection with $P$. You can't just look at $P$ because you need trannsversality, but you want only a tiny neighborhood of $P$.

So you get $H_*M \to H_*(Thom(\nu)) \to H_{*-\dim \nu}P$. If $[Q] \in H_*M$ then its image will be $[P \cap Q]$.

Now I'll do this in the case of the loop space instead of the manifold. Instead of $P$, consider maps of a figure eight into $M$. This is a subset of $LM \times LM$. We want some intersection "with this map space Maps$(\infty, M)$." So we want $H_*LM \times LM \to H_*\mathrm{Maps}(\infty, M)$. Consider the following. There's an easy way to look at two loops and see if they can make a map from the figure eight. So you get the pullback square

$$\begin{tikzcd}
\mathrm{Maps}(\infty, M) \arrow[r, hook, "\rho"] \arrow[d, "ev_\infty"'] & LM \times LM \arrow[d] \\
M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

References include Cohen-Jones, Homotopy theoretic realization of string topology and Cohen-Voronov, "Notes on String Topology."

This square is infinite dimensional on top, but finite on the bottom. The codimension is just the dimension of $M$. So $\rho$ has finite codimension. We can go to a tubular neighborhood of $M$ so we can pull back and build the Thom collapse map upstairs.

- The bottom line is an embedding of finite dimensional manifold, so we have a tubular neighborhood.
- What we get is the pullback of the tubular neighborhood of $\Delta$ is a neighborhood of $\rho$, which still needs to have an identification with the normal bundle.

You can do this by replacing it with the path fibration, then lift, you have a vector at the end of the path, you can push the path through the vector, in this case it's locally trivial so it just works, but [some discussion]

You get a tubular neighborhood eventually and then a Thom class since $\nu_\Delta \cong TM$. So you get $H_\alpha LM \times LM \to H_*Thom(ev^*TM) \overset{\to}{Thom} H_{*-d}\mathrm{Maps}(\infty, M)$. Define the loop product by

$$H_*LM \otimes H_*LM \xrightarrow[\rho!]{} H_{*-d}\mathrm{Maps}(\infty, M) \xrightarrow[\text{composition}]{} H_{*-d}LM$$

Remark. By construction, $(H_*M, \wedge)$ goes by the constant maps to $(H_*LM, \bullet)$ which goes by intersection to $(H_{*-d}\Omega M, \text{Pontryagin product})$ These are algebra maps.

[Why is this wrongway?]

The functor gives you a map the wrong way. This is why Poincaré duality would help, you can move to cohomology to get the map to go the other way.

**Theorem 6** *Chas-Sullivan*
*$\bullet$ is associative and graded commutative.*

I'm not going to worry about signs, the equations get confusing. Associativity, I'm not going to do it formally. It's easier with a picture. Consider if you have three transverse chains $x, y, z$, if you consider what you want to do to define $(x \bullet y) \bullet z$ and $x \bullet (y \bullet z)$, the first thing is where is this defined? In the first case, you want $x, y$ to start at the same point, and then that point must be the same point as the start of $z$. So in both cases, they are defined on $(\vec{s}, \vec{t}, \vec{r})$ such that $ev(x(\vec{s}) = ev(y(\vec{t}) = ev(z(\vec{r})$ So for any $(\vec{s}, \vec{t}, \vec{r}) inA$, you compare $(x(\vec{s}) * y(\vec{t})) * z(\vec{r})$ with $x(\vec{s}) * (y(\vec{t}) * z(\vec{r}))$.

There is a notural homotopy between these two chains. This shows why it should be associative. This just takes the result for based loops and shows why it should work in families. It's more subtle and interesting to consider commutativity.

Instead of just looking at paths from the figure eight with the basepoint, you can let this vary over the circle. Instead of $\mathrm{Maps}(\infty, M) \to LM \times LM$, look at $P \subset LM \times LM$ with $(\alpha, \beta, t)$ so that $\alpha(0) = \beta(t)$. You go around $\beta$ for a bit, move onto $\alpha$ at $t$ and then finish $\beta$ after $\alpha$.

So you have a pullback square

$$\begin{tikzcd} P \arrow[r, hook, "\rho"'] \arrow[d, "ev"'] & LM \times LM \times I \arrow[d, "ev_0 \times ev_t"'] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd}$$

As before we get $C_pLM \otimes C_qLM \otimes C_1I$ by $\rho!$ to $C_{p+q-d+1}P \to C_{p+q-d+1}LM$ So $\star : C_pLM \otimes C_qLM \to C_{p+q+1-d}LM$ by $\otimes[I]$.

I claim that $\partial(x \star y) = \partial x \star y \pm x \star \partial y \pm x \bullet y \pm y \bullet x$. So up to boundary being zero you get $x \bullet y = \pm y \bullet x$.

The proof is $\delta(x \otimes y \otimes [I]) = (\delta x) \otimes y \otimes [I] \pm x \otimes \delta y \otimes [I] \pm x \otimes y \otimes 1 \pm x \otimes y \otimes 0$. So $x \otimes y \otimes 0$ you go around $\alpha$ first whereas at 1 you go around $\beta$ first.

Okay, and part of the fun and the beauty of the argument is that the figure eight was arbitrary, you can break it up with intersections on something else. You could use intersections of bigger diagonals $\Delta : M^S \to M^T$. All of this comes out of the proof of commutativity, well, out of the construction. I'm going to stop.

# 3 McClure, $A_\infty$ operads

Now we have Jim McClure, He'll be talking about $A_\infty$ operads.

Here are some references:

McClure-Smith: Operads and Cosimplicial objects, an introduction (sections 2, 6, 9). This overlaps somewhat with this, it's available on the arXiv.
Markl-Shnider-Stasheff. Finish this book, get in touch with me and I'll give another reference.

Operads, the definition makes people nervous so I'll go slow. The motivating question, is, given $Y$ is there a $Z$ with $Y \cong \Omega Z$. (weakly homotopic). This was the late 50s and early 60s. In the first lecture, you wanted to answer this question.

[Weakly homotopic? A map between them is an isomorphism on every homotopy group]

What structure does $\Omega Z$ have? Then we can see if $Y$ has the same structure. If you learn enough about $\Omega Z$ you can do this.

It's well known that $\Omega Z$ has a homotopy associative multiplication, not strictly associative but homotopy associative. There used to be an industry to create such spaces, this is not enough.

I'll give the slogan, then use the hour to explain it. The slogan is, we need "higher homotopy associativity."

To get a hold an this higher associativity, the key idea is to consider multivariable and not just binary operations. Okay? So that's the key, that's what operads are all about. We want to go beyond multiplication.

As a digression, what happens at the set level? Let's practice. This is the basic thing, composition, that everything else is built on. Soy $f : T^k \to T$, for $T$ a set. Now say I have maps $g_i : T^{j_i} \to T$ for $i \leq k$. Then you can get $f(g_1, \ldots, g_k) : T^{\sum j_i} \to T$. This is amenory, but it's good to get warmed up.

So now let me discuss strict associativity. This is the relation between strict associativity and multivariable composition. This is a fancy way of saying what a monoid is.

**Proposition 2** *A monoid structure on a set $T$ is equivalent to a sequence of maps $M_k : T^k \to T$ for $k \geq 0$ such that*

*a.* $M_1 = id$

*b.* $\{M_k\}_{k \geq 0}$ *is closed under multivariable composition.*

This is the same as putting a strictly associative structure with unit on $T$. If you have a monoid, let the identity be $M_0$, the $n$ to 1 multiplication $M_n$.

How do you go the other way? How do you know you have associativity? You want these to commute:

$$\begin{tikzcd}
T \times T \times T \arrow[r, "M_2 \times M_1"] \arrow[d, "M_1 \times M_2"'] \arrow[dr, "M_3"] & T \times T \arrow[d, "M_2"] \\
T \times T \arrow[r, "M_2"'] & T
\end{tikzcd}$$

Now we want to look for sumilar structures on $\Omega Z$. You have basic maps for all $r \in (0,1)$ you get a multiplication $\mu_r : \Omega Z \times \Omega Z \to \Omega Z$. There's no natural choice, if you pick $\frac{1}{2}$ it's not associative.

Let $* \in Z$ be the basepoint. Then $e : (\Omega Z)^0 \to \Omega Z$ is the constant map, that's the unit here.

I need something closed under multivariable composition, which this is not. For example, I can take the interval, break it into pieces of arbitrary lengths, and have a bunch of paths with maps to the basepoint in between them. This is a map from $(\Omega Z)^k \to \Omega Z$. This is built from these ingredients.

**Definition 1** *$\mathscr{A}(k)$ (the little intervals operad) is the space of all maps from $(\Omega Z)^k \to \Omega Z$ obtained from $e$ and $\mu_r$ from multivariable composition.*

That condition before was a way of saying strict associativity, now it will be higher homotopy associativity.

Note that $\mathscr{A}(k)$ has an abstract description which does not refer to $Z$. A point in $\mathscr{A}(k)$ is a collection of $k$ closed subintervals of $[0,1]$ with nonintersecting interiors.

As an exercise,

**Exercise 1** *Write, give the explicit description of the multivariable composition in the collection $\{\mathscr{A}(k)\}_{k \geq 0}$ with this description.*

**Exercise 2** *$\mathscr{A}(k)$ is contractible*

What have we shown now?

[Is this labelled?]

This is a non-$\Sigma$ operad.

[Do you need $e$?]

Anything outside the intervals goes to the basepoint.

The goal was to find the structure on the loop space

**Proposition 3** *If $Y = \Omega Z$ for some $Z$ then there is a sequence of subspaces $\mathscr{A}(k) \subset$* $\mathrm{Maps}(Y^k, Y)$ *such that* *a.* $\mathscr{A}(1)$ *contains id*

*b. the family* $\{\mathscr{A}(k)\}_{k\geq 0}$ *is closed under multivariable composition,*

*c. each* $\mathscr{A}(k)$ *is contractible.*

This is like a homotopy version of what happened before, a point is replaced with a contractible space.

Points are homotopic, but then the homotopies (paths between them) are homotopic, and the homotopies of paths are homotopic, and this is what we mean by higher homotopy associativity. We didn't need to say the associativity explicitly.

This is a converse of sorts.

**Proposition 4** *(preliminary version)*
*If* $Y$ *is connected and there is a sequence of subspaces* $\mathscr{O}(k) \subset \mathrm{Maps}(Y^k, Y)$ *such that*

*a.* $\mathscr{O}(1)$ *contains id*

*b. the family* $\{\mathscr{O}(k)\}_{k\geq 0}$ *is closed under multivariable composition,*

*c. each* $\mathscr{O}(k)$ *is contractible.*

*then there exists* $Z$ *such that* $Y$ *is weakly homotopic (eventually homotopy equivalent) to* $\Omega Z$*.*

So I've answered my initial question, why don't I stop here? How do you find $Z$? With the bar construction, I'll discuss that more later. Before I get to this I want to give a more general version. This motivates a preliminary definition of operads. But it's awkward. Operads allow you to permute the index. You include the $\Sigma_k$ action. So this is a preliminary definition of a non-$\Sigma$ operad. It's like smoking or non-smoking.

[If you compare it to smoking it's only a matter of time.]

**Definition 2** *(preliminary)*
*A non-*$\Sigma$ *operad* $(O)$ *is a collection of subspaces* $\mathscr{O}(k) \subset \mathrm{Maps}(Y^k, Y)$ *such that*

*a.* $\mathscr{O}(1)$ *contains id*

*b. the family* $\{\mathscr{O}(k)\}_{k\geq 0}$ *is closed under multivariable composition.*

*If the spaces are contractible you can call it* $A_\infty$*.*

[Does Y matter?]

Let me, well, people also get confused on this next abstracting step. Let me remind you, in a digression, what the 19th century definition of group was. It was a family of transformations of a set $S$ containing the identity map and closed under composition and inverses.

[Now I feel silly asking if $Y$ matters]

Good, that's my goal, to make you feel silly.

So we split this, well I didn't, but it was split into the notions of abstract group and a group action. I could just leave the parallel construction for operads as an exercise right now, but I'm going to give hints.

So first we want the definition of an abstract non-$\Sigma$ operad. Note that we have multivariable composition maps $\mathscr{O}(k) \times \mathscr{O}(j_1) \times \cdots \times \mathscr{O}(j_k) \to \mathscr{O}(\sum j_i)$.

So suppose I have $f, g_1, \ldots, g_k$, i'll just say this maps to $f(g_1, \ldots, g_k)$. So what does it mean for this to be associative? This is a stumbling block because it's a little complicated. If I take $f(g_1(h_{11}, \ldots, h_{1j_1}), \ldots, g_k(h_{k1}, \ldots, h_{kj_k}))$. There are two ways of simplifying this. I can do first the $g$ then the $h$ or the other way. They should be equal.

**Exercise 3** *If you write this out correctly as a commutative diagram, you should get the first diagram of Peter Mays' book Geometry of Iterated Loop Spaces.*

I'm ruuning out of board space. You need more conditions. You want $f(id, \ldots, id) = f$ and $id(g) = g$.

**Definition 3** *An abstract non-$\Sigma$ operad $\mathscr{O}$ is a collection of spaces $\mathscr{O}(k)$ for $k \geq 0$ with ("composition") maps $\mathscr{O}(k) \times \mathscr{O}(j_1) \times \cdots \times \mathscr{O}(j_k) \to \mathscr{O}(\sum j_i)$ and an element in $\mathscr{O}(1)$ called id satisfying the associativity and identity relations above.*

*If all of the spaces $\mathscr{O}(k)$ are weakly equivalent to a point we say $\mathscr{O}$ is $A_\infty$.*

The next goal should be to define an action of a non-$\Sigma$ operad.

In the preliminary definition we had evaluation maps. Her we get maps $\mathscr{O}(k) \times Y \to Y$. So this takes $f, y_1, \ldots, y_k \to f(y_1, \ldots, y_k)$.

**Exercise 4** *Figure out the analogs of the relations on an abstract operad.*

**Exercise 5** *Make this a commutative diagram, it will be another in the beginning of the Mays book.*

**Definition 4** *Let $\mathscr{O}$ be a non-$\Sigma$ operad, $Y$ be a space. An action of $\mathscr{O}$ on $Y$ is a sequence of maps $\mathscr{O}(k) \times Y^k \to Y$ satisfying the conditions I didn't write down.*

**Theorem 7** *(Improved version of the third proposition) If $Y$ is connected and an $A_\infty$ non-$\Sigma$ operad acts on $\pi$ then $Y$ is weakly equivalent to $\Omega Z$ for some $Z$.*

*In fact you can replace "connected" with "grouplike" where $\pi_0(Y)$ is a group with respect to $\mu : Y \times Y \to Y$ and the induced $\mu_* : \pi_0 Y \times \pi_0 Y \to \pi_0 Y$. This makes the theorem necessary and sufficient.*

# 4 Discussion

[Are there natural examples that don't arise from this construction.]

Yes, there is one. It's not quite right that it doesn't arise, it doesn't come naturally. It comes from the action of a parameter on the loop, it's the "Thom twist." Ordinarily you like to look at the trivial $\underline{\underline{Y}}$-bundle $Y \times X = \Omega^d Y^d \times X$. There is a notural way of looking at this involving looking at $\mathrm{Maps}(Dv, \delta Dv, Y^d) \to X$. This is also a map from $(DV, \delta Dv) \to Y^d$. This creates somethnig, defines $h^d_Y(DV, \delta Dv)$. This defines a twisted class ${}^{\tau(v)}h^0_Y(X; \underline{\underline{Y}})$. Then the Thom isomorphism ${}^{\tau}vh^0_Y(X) \cong h^d_Y(Dv, \delta Dv)$. This is an example that doesn't come from an action of the spaces on the loops.

If $d = 1$ you get a real line bundle, which is in bijection with the double cover $\tilde{X} \to X$. So this gives the $\pm 1$ action. When you write $Y = \Omega Y^1$, reflection is the inverse map. So twisting by this kind of isomorphism is kind of a Thom twist. Does that answer your question?

[Some asked "where are the twists?" Where did the idea come from?]

In $K$-theory it was thought of in connection with Brouwer groups of algebras. You can talk about it for algebras or projective modules, nad then you ask if you have a twisted matrix algebra, after an isomorphism theory about matrix algebras (Morita). This led to the study of twisting of the $K$-theory over a space instead. This happened when physicists said boundary states have something to do with string, brane, blah, [unintelligible]. This was a compelling answer.

I study it because I had a statement which had an algebraic side but not a topological side, for loop groups and projective representations.

[Can you tell us about spinors?]

You'll have to be more specific. The construction of the Thom class, let me start slightly easier, if $V$ over $X$ is a complex vector bundle one can construct the Thom class in $K^{2d}(Dv, \delta Dv)$. This restricts to the generator of $K^0(Dv_*, \delta Dv_*) = K^0(S^{2d}$ in each fiber. Then multiplication by $\theta$ defines an isomorphism $K^*(X) \to K^{*+d}(Dv, \delta Dv)$.

[Excised]

Let $E^0 = \oplus_{k\ even} \wedge^k V$ and $E^1 = \oplus_{k\ odd} \wedge^k V$. Then the isomorphism at $v \in V_x$ at $x$ is $v + i(v)$. $E^0 \to E^1$ squares to $2||v||^2$ this is an isomorphism on $D_v$ away from the zero section. A fact is that [unintelligible]generator are $K^0$ of each sphere ($\pi_{2d}BU$) for $S^2$ it's the Bott map.

If you give up the integer grading on $\wedge^{\cdot} V$ and keep only mod 2 grading, you need less than a complex structure. You need a spin structure. You want to improve the Thom isomporphism to things with somewhat less structure.

**Definition 5** *Let $V$ be a real even dimensional vector space, say it's complex. If $b$ is a nondegenerate bilinear form the Clifford algebra $Cliff(V)$ is generated by $V$ with relations $\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = 2b(v, w)$. This is graded mod 2, we declare $\gamma(v)$ to be odd and 1 even.*

**Theorem 8** *$Cliff(V)$ with its frading is $End(S^0 \oplus S^1)$. This is $\left(\begin{array}{c|c} even & odd \\ \hline odd & even \end{array}\right)$. This has only one irreducible module up to isomorphism. So any automoprhism gives an automorphism of $S$ up to phase.*

Now $SO(V)$ acts "projectively" on $S^0 \oplus S^1$. $\pi_1 SO = \mathbb{Z}_2$ so the phase ambiguity is $\pm 1$. It's there, the double cover sits inside $Cliff(V)$, $Spin(V)$ of $SO(V)$. Given $V \to X$ can form a bundle $Cliff(V)$ of matrix algebras. Assume that $S_V/S^0_V \oplus S^1_V$ "exists." This is a 1-cocycle with values in $SO$. SO we get $\to H^1(X; SO) \to H^2(X; \mathbb{Z}_2) \ni w_2(V)$, the second Stiefel-Whitney class.

Define $\theta in K^0(Dv, \delta Dv)$. So $E^0 = S^0,\ E^1 = S^1$, at $v \in V^*$, let the map $S^0 \to S^1$ be $|gamma$ then $\gamma(v)^2 = ||v||^2 \neq 0$ as a 0-section.

$$\begin{tikzcd}
H^2(X;\mathbb{Z}_2 \arrow[r] \arrow[dr] & H^3(X,\mathbb{Z}) \arrow[d, leftrightarrow] \\
 & H^2(X;U(1)
\end{tikzcd}$$

Then $w_2(V) \to w_3(V)$.

[At the beginning, you extend to positive by choosing a $Y$. Does the choice matter?]

Definitely, there are things you can choose at any level to change it, it may not be periodic.

Tomorrow we're going to discuss the part of $K(\mathbb{Z}, 2) \hookrightarrow Aut(K)$. $GL_1^+(K)$ is not a topological group but is an infinite loop space. something about $K(\mathbb{Z}, 2) \times BSU_{\otimes}$.

### 4.1 String topology

[I have a stupid question, what is a pullback square?]

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & A \arrow[d] \\ C \arrow[r] & B \end{tikzcd}$$

If you have a map from something into $A$ and $C$ then you get a unique map into ?. If you take

$$\begin{tikzcd} & LM \times LM \arrow[d] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd}$$

and you want to fill it in you get

$$\begin{tikzcd} \mathrm{Maps}(\infty, M) & LM \times LM \arrow[d] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd}$$

This is a fibration, so you get that it is a homotopy pullback.

You said you could do this for other figures. The first generalization of this was by Cohen-Jones, using the little disk operad, you look at a bunch of circles and you have a way of composing these, it's more, you have an operad, you can substitute any of these diagrams for each circle.

You can go further to fat graphs, a graph with a cyclic ordering at the vertices. In some sense it's that extra data that tells you how to go through your graph to make circles. You can then pick circles out by starting along an edge and moving to the next edge at each vertex. On different circles you get different powers of $LM$ as the target from $\mathrm{Maps}(\Gamma, M)$.

This graph can be thickened by preserving the cyclic order, replacing edges with thin strips. You need to know the cyclic ordering or you can get a different surface.

The ones where you can do the wrongway map it's $H_*LM^P \to H_{*-\chi(\Gamma)d}\mathrm{Maps}(\Gamma, M)$.

[The homology of the moduli space of these graphs, it's the same as the moduli space of the surfaces?]

No. The goal is to get the homology of the moduli space to parameterize operations.

Kauffman claims to have it, I didn't see it on the arXiv. Sullivan has something, but he's compactifying or something.

[...]

The conjecture is if $MCG(s,\delta) = \pi_0(Diff^+(S,\delta))$ $H_*MCG(S,0) \otimes (H_*LM)^{\otimes p} \to H_*LM^{\otimes q}$. This didn't work the way it was before, because the space was not a high enough dimension.

[Ralph knows how to do it on the Hochschild complex of a Frobenius algebra, but he can't do it on a manifold yet.]

[You have a model that gives you a moduli space, and for each graph you have a wrongway map?]

Take the graph with two lollipop ends on an interval. Take maps on this on $LM \times LM$. You don't get an embedding. If you take a $\theta$ instead, then you get an embedding, but you need the two loops to agree on an entire path, which is infinite codimension. In the first case you can add a $PM$ factor to make it an embedding, you just need to choose.

[So what can you say about how this relates to the [unintelligible]operad? If this were to work would that make the appropriate thing an $A_\infty$ algebra?]

Where's Sorin, didn't you ask me that? When the propagator is invertible, ...

[A bunch of stuff I couldn't follow?]

[I did not understand the last claim, to why $x \otimes y \otimes 1 \to x \bullet y$. The map is from $P = \{(\alpha, \beta, t), \alpha(0) = \beta(t)\} \subset LM \times LM \times I$. Then $C_*(LM \times LM \times I) \to C_*(P) \to C_*(LM)$. So if you take $t = 1$, you are taking two loops based at one point and moving around one then the other. If you start with transverse chains, the shriek map does this obviously, doing it directly, that could be difficult.

Part of this is doing the construction over chains, passing from one to the other by moving one around the other. If you want to do it formally, you can do it over the entire thing. You would get exactly that along the boundary.

[I just understood what you said. I thought that the homology was semisimple, then you can pass to the spectrum. That's why it was a very clean answer.]

[Other questions? Then we will have Jim without a break.]

[Could you explain [unintelligible]]

There should be a framed little disks operad action that induces the BV structure. The little disks gives you a Gerstenhaber algebra. If the bracket is nonzero that's the obstruction of getting to the framed 3-disks.

[What was the last thing?]

$\pi_0$ of a loop space is a group. To characterize loop spaces you have to build that in somehow. If you have an $A_\infty$ operad acting on a space. Call it $\mathscr{O}$ acting on $Y$. Then choose $\mu \in \mathscr{O}(2)$. We know $\mathscr{O}(2) \times Y \times Y$ maps to $Y$. So a point maps to $* \times Y \to Y$ which then thusly maps to $Y$. This induces $\mu_* : \pi_0(Y) \times \pi_0(Y) \to \pi_0(Y)$. This makes it a monoid. In the little interval case this would be a group.

Any other $\mu$ would be homotopic by a path in $\mathscr{O}(2)$.

**Definition 6** *The action of $\mathscr{O}$ on $Y$ is grouplike if $\mu_*$ makes $\pi_0(Y)$ a group.*

[It's a more general question, can you explain the link between all that stuff and string theory?] No. That was a really easy question.

[Is there a slick way to state the group completion?]

That's part of the proof. If you have the $A_\infty$ structure you can define $B(Y)$ and then you use group completion to go on from there.

[Let's thank Jim. We'll meet tomorrow then.]