Stringy Topology Notes
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1 Uribe, K-theory

For those of you who have seen this I'm following Atiyah's book on K-theory. A vector bundle $E \to^{\pi} X$ has $\pi^{-1}(\{x\}) = Ex \cong \mathbb{C}^n$.

These will all be complex for me. It must be locally trivial, around every point there is a neighborhood U with $E|_U \cong \mathbb{C}^n \times U$.

Examples include the tangent space of a complex manifold and $\mathbb{C}^n \times X$.

It can also be given by local information. On $U_\alpha \cap U_\beta$ there is a map $g_x : \mathbb{C}^n \to \mathbb{C}^n$ in $GL(n, \mathbb{C})$ which takes one trivialization to another. We have $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = id$.

There is a functor from this category to the group category of $GL(n, \mathbb{C})$ where g_x goes to the matrix of g_x.

These have all the properties you can imagine. If you have two different vector bundles you can $\oplus, \otimes,$ $Hom, \wedge^*$. Whatever you can do to fibers can be done in a global sense. You can pull back

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"'] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

What is a morphism of vector bundles? It's a continuous map which is linear on the fibers.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & F \arrow[dl] \\ X & \end{tikzcd}$$

Here are some results I'm not going to prove. If $f : X \to Y$ is a homotopy equivalence then f^* which takes isomorphism classes of vector bundles over X to classes over Y is bijective.

If X is contractible, then as a corollary, the class of vector bundles over X is the same as over a point, so it is $\mathbb{N}$, just the rank.

Lemma 1 $Vect_n(SX) \cong [X, GL(n, \mathbb{C})]$

The suspension is the cone of X with the base contracted. How do you cover it? You can cover it with two cones C^+ and C^-. The vector bundles on these are trivial. You define a bundle on SX by gluing two trivial bundles. This gluing is done along X. You restrict $E \to SX$ to $C^+ \cap C^-$, and this is just the transition map. So this is a map from $C^+ \to C^-$ to $GL(n, \mathbb{C})$. This is a generalization of what happens with the sphere, which is a suspension of a lower dimensional sphere.

1.1 Grassmanian

What is the Grassmanian? $Gr_n(\mathbb{C}^m)$. These are n-dimensional vector spaces in $\mathbb{C}^m$. For example $Gr_1(\mathbb{C}^{m+1}) = \mathbb{P}^m$.

Over the Grassmanian you have the tautological bundle. $\gamma_{n,m}$ has over every point of $Gr_n(\mathbb{C}^m)$ the vector space that determines it.

You have $\mathbb{C}^m \hookrightarrow \mathbb{C}^{m+1} \hookrightarrow$ and so you can get $Gr_n(\mathbb{C}^\infty) = \varinjlim Gr_n(\mathbb{C}^m)$.

You have a free action of $GL(n, \mathbb{C})$. Now $\gamma_n \to Gr_n(\mathbb{C}^\infty)$ which is $BU(n)$.

A theorem: If X is paracompact, then $[X, Gr_n(\mathbb{C}^\infty)] \to Vec_n(X)$ is bijective. This is given by

$$\begin{tikzcd} f^*\gamma_n \arrow[d] & \gamma_n \arrow[d] \\ X \arrow[r, "f"] & Gr_n(\mathbb{C}^\infty) \end{tikzcd}$$

Remark. If E is a vector bundle over X and $\Gamma(E)$ a global section ($\pi \circ \Gamma = id$) then $\Gamma(E)$ is a module over $C(X)$, global functions $X \to \mathbb{C}$. The trivial bundle is a free $C(X)$-module and $Vect(X)$ are the finitely generated projective modules.

Now I can define K-theory. $K(X)$ (which is the K-functor on the semigroup of vector bundles). This is pairs of vector bundles $([E], [F])$ which we should think of as $[E] - [F]$. I don't want to keep writing these brackets. This is under the relation $([E], [F]) \sim ([E'], [F'])$ if there exists G with $E \oplus G \oplus F' \cong E' \oplus G \oplus F$.

Now for any vector bundle E (this is a lemma) over compact X there exists F with $E \oplus F$ trivial. You have to prove this.

We will write $E \oplus F = n$ where n is the rank of the trivial bundle. So we can write $([E],[F])$ as $([H],n)$.

Lemma 2 *$K(X) \cong \mathbb{Z}]\mathbb{Z} \times \varinjlim Vect_n(X)$ where the direct limit is from the embedding $E \mapsto E \oplus 1$, the trivial line bundle.*

Please ask me if you have questions.

Define U as the direct limit of the unitary groups $U = \varinjlim U(n)$. Here $U(n) = \{A \in GL(n,\mathbb{C}) | AA^* = 1\}$. We say $BU = \varinjlim BU(n)$. If I take $[X, BU]$ this is, you have to be careful about pulling out limits, this is $\varinjlim [X, BU(n)]$ which is $\varinjlim Vect_n(X)$, which is called the reduced K-theory $\tilde{k}(X)$. Ernesto wrote yesterday $K(X) = [X, \mathbb{Z} \times BU]$.

Now let me write $\tilde{K}(SX) = [SX, BU] = [X, \Omega BU]$, where this is the space of based loops, which is $[X, U]$. So the K-theory of the suspension is to just take the equator, and then the transformation group, which is what we had before.

Now let me just define the relative groups. I haven't said it, I can add and tensor these. Take $Y \subset X$ with everything compact for simplicity. $(X,Y) \rightsquigarrow X/Y$, which has the basepoint Y/Y. This gives me K-theory with a basepoint.

So think of $X \rightsquigarrow X^+ = X \sqcup *$. This is so the long exact sequence works well. There is a map $K(X^+) \to K(*)$, which takes everything to the integers, so the kernel is basically the K-theory of X, which is the reduced K-theory of X^+.

This allows me to define, for based point sets X, Y, the smash product $X \wedge Y = X \times Y / X \times \{y_0\} \cup \{x_0\} \times Y$ Here we can see $\wedge^n S^1 = S^1$ and $S^n(X) = S^n \wedge X$. Now

Definition 1 *$K^{-n}(X) = \tilde{K}^{-n}(X, \emptyset) = \tilde{K}(S^n(X^+))$ where $K^{-n}(X,Y) = \tilde{K}^{-n}(X/Y) = \tilde{K}(S^n(X/Y))$.*

Lemma 3 *$i : Y \to X, j : (X, \emptyset) \to (X,Y)$ then*

$$0 \longrightarrow K(X,Y) \xrightarrow{j_*} K(X) \xrightarrow{i_*} K(Y) \longrightarrow 0$$

is short exact.

I am going to use this for a long exact sequence. Take the mapping cone $X \cup CY$ and then $X \cup CY/X = CY/Y$ so $J(X \cup CY, X) = \tilde{K}(CY/Y) = \tilde{K}(SY) = \tilde{K}^{-1}(Y)$.

The above short exact sequence works with tildes so I get a long exact sequence

$$\longrightarrow K^{-n-1}(Y) \longrightarrow K^{-n}(X,Y) \xrightarrow{j_*} K^{-n}(X) \xrightarrow{i_*} K^{-n}(Y) \longrightarrow$$

Bott Periodicity says $K^{-n}(X) \cong K^{-n-2}(X)$ because $\Omega^2(U) \cong U$.

So the long exact sequence becomes exactly

$$\begin{tikzcd} K^1(Y) \arrow[d] & K^1(X) \arrow[l] & K^1(X,Y) \arrow[l] \\ K^0(X,Y) \arrow[r] & K^0(X) \arrow[r] & K^0(Y) \arrow[u] \end{tikzcd}$$

No what about $K(S^2)$? Well $K^0(S^2) = \mathbb{Z} \oplus \mathbb{Z}$ and $K^1(S^2) = 0$. Onne $\mathbb{Z}$ is from the trivial bundle and the other is from the Hopf fibration $\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[d] \\ & S^2 \end{tikzcd}$.

Another useful way to see K-theory is with Fredholm operators. Let $\mathscr{H}$ be a Hilbert space and then let $o(\mathscr{H})$ be bounded operaters. Then $T \in o(\mathscr{H})$ is Fredholm if the kernel and cokernel are finite. The index $ind(T) = \dim\ker(T) - \dim\ \mathrm{coker}(T)$.

Let $\mathscr{F}$ be the space of Fredholm operators. This is closed under composition. If I take $[x, \mathscr{F}]$ (norm topology) this is a semigroup. Then you can take the index of the operator.

Theorem 1 *(Index theorem)*
$Ind : [x, \mathscr{F}] \xrightarrow{\cong} K^0(X)$.

Locally this is a difference of vector spaces. You have to fiddle a little to keep things the same dimension.

Let me finish with Chern classes. We want to measure these things. I have a classifying space $BU(n)$. What is the cohomology? I take a maximal torus $T^n \subset U(n)$. Up to blah blah blah I have a map $H^*(BU(n)) \to H^*(BT^n) = (H^*(BS^1))^n = \mathbb{Z}[x_1, \ldots, x_n]$ where x_i has degree two. The image is polynomials invariant under the Weyl group. $H^*BU(n) \cong (H^*BT^n)^W = \mathbb{Z}[x_1, \ldots, x_n])^{S_n} = \mathbb{Z}[c_1, \ldots, c_n]$. Here c_i is the ith Chern class.

$$\mathbb{C}^n \longrightarrow E$$

$$X \xrightarrow[f]{} BU(n)$$

Here $C_i(E) = f^*C_i \in H^{2i}(X, \mathbb{Z})$ and $C_t(E) = \sum t^i C_i(E)$. We have $C_t(E \oplus F) = C_t(E)C_t(F)$ and $ch : K^*(X) \to H^*(X, \mathbb{Z})$ where $E \mapsto \sum e^{x_i}$. Now $Ch_{\otimes\mathbb{Q}}$ is an isomorphism so up to torsion they measure the same thing.

2 Lupercia, Physics to Algebra

Thank you very much, today we have a lot of ground to cover. Everything today will be extremely explicit and elementary. The point is to show how physical systems lead to algebraic systems.

Let me start with a well-known problem in physics, variational problems.

[Excised]

3 Discussion

We have the excursion for Saturday. The announcement is on the board. It will be on Saturday. We'll depart at 9:00 or 9:30, it will be about 120 Pesos. It's a charming town on a lake, with an island in the middle. You can walk around it and so on, eat a lot, drink a bit, go back to the town. Before that to an old indigenous site, with pyramids. We'll have an English-speaking guide.

Ernesto hasn't come, so if you have questions about what he said, sorry. He said to have a symplectic structure ω in a manifold was the same as saying that $H^{2k}(M,\mathbb{R}) \neq 0$. This is conjecture one.

Here's one direction. If M is symplectic then ω is a 2-form with $d\omega = 0$ and $\wedge^n\omega$ is a volume form. So if M is compact, then this says that $\omega^n \neq 0$ in $H^{2n}(M,\mathbb{R})$. So $[\omega]^k \neq 0$ for $1 \leq k \leq m$.

The trivial observation if M is noncompact, is take $\mathbb{R}^{2n}$, the cohomology vanishes but it's still symplectic.

Conjecture one is false, $S^2 \times S^4$. This has nonzero homology in every even dimension. Take the only one in dimension two, and its product with itself is zero.

Next if M is compact and there exists $\alpha \in H^2$ such that $\alpha^n \neq 0$ in H^{2n} then M is symplectic. This is conjecture two, it should be false too. If M is symplectic then the tangent bundle of M admits a complex structure, that is, M is almost complex. So $J : TM \to TM$ has $J^2 = -id$. This imposes conditions which are not present in cohomology.

I don't have an elementary example of such a manifold. I believe you can do it by thinking about Pontryagin classes. By a theorem of Taubes, whenever X is a 4-manifold with a class in H^2 satisfying $\alpha^2 > 0$ in cohomology, using Seiberg Witten theory, $X\#\mathbb{CP}^2$ is not symplectic. If I said this for $X\#\overline{\mathbb{CP}}^2$ this would be false, this is just blowup.

Gromov proved the following in the seventies. A smooth open manifold M admits a symplectic structure if and only if it is almost complex.

[The Lie bracket has a geometric interpretation, does the one for polyvector fields?]

Yes, it does, it is extremely involved. You suppose that there are integrable surfaces. How many zeroes of one does another have. I could look through my notes but they are complicated.

Anything else? This is like calculus class, everything is clear?

You can measure the obstruction to an almost complex structure? It has something to do with Pontryagin classes.

Ernesto, you explained how the third Hochschild cohomology controls deformations. The higher ones?

They are related to A^∞ algebras. I don't know about how they control deformations. You may want to look at the reference by Sasha Voronov, or Robinson's work on putting these structures on different things.

[Can I ask you do one of your exercises? Hochschild as extension group?]

No, you do it. It's in absolutely every book, including Eilenberg Cartan.

One piece, I hope these discussion sessions will work. Otherwise people will think, why do we do these things? At least to get exact references.

I have a naive question, you gave a concrete presentation of how Poisson structures arrive. What physics generates the other kinds of algebra?

Instead of a symplectic manifold you can have a Poisson algebra. Then we had the story with the polyvector fields and instead of these we decided to think of the Gerstenhaber algebra. The connection on the polyvector fields led to BV algebras. These algebras can be thought of in a different way. The connections on polyvector fields are close to physics, but this is not really how physicists would approach this. They think of quantum not classical systems. But the quantizations can still be thought of with algebras of this type. The algebras they talk about are algebras of operators of a quantum system. I didn't want to get into a longer discussion because that would be too long. I wanted to point out that they appear even at the classical level, we just didn't see them that way because we weren't looking.

This is a typical trend, the example is the concept of supermanifolds. It seems unmotivated at first. It was discovered at the quantum level, and the mathematician asked what sort of classical object led to this quantization. One realizes sometimes that we didn't even know of whatever the classical thing was.

Other questions?

[This connection identifying polyvector fields with the Hochschild cohomology. There's also Deligne's conjecture showing it's a BV-algebra if it's there's a Frobenius structure. Are these the same?]

Yes. They do not make conjectures like this without reason.

[What is the definition of the Bott element?]

The Bott element, the story is the following, morally the story is that somehow you want $K(X \times S^2) = K(X)$. Maybe with smashes or this or that, I will ignore it. The Bott element is $[H] - 1$ (this is the Hopf fibration). The proof is very simple. If you have S^2 with X on it at the points, with S^2 big and X small, you consider $X \times S^1$, if I remember what you do, you consider the Weyl operator on S^2, then you have some sort of, well, if X is a point, you do it in families. If you have a vector bundle, you twist, tensor, with E and then go via index to $\mathbb{Z}$. You have a family which leads to a vector space. And then by the formal properties you can check that this and a similar one are inverses. You need the Riemann Roch theorem, for one calculation with the Hopf fibration.

What's the connection between the BV and Gerstenhaber algebras from physics and the ones from the little disks operad?

Jim is going to talk about this stuff. Berdina will also get another relation between these subjects.

[A Gerstenhaber algebra that is not BV?]

I don't know. It's like a guy, a guy can be naked or can wear different suits, different BV structures, is there a guy who goes around naked, no clothes in the world fit him? I don't know, there must be such a guy, I don't know.

[The moduli space of connections on Gerstenhaber algebras?]

I don't know. Someone does. You should ask it yourself. Other questions? Did you do all the problems? Were we understandable or were we obscure?

[You were a little fast.]

Thank you.

[That wasn't a compliment.]

I mean thank you for the remark. I want to prepare you for the really fast talkers. Konstantin asked if this was for graduate students. I said yes, don't tell him, so that I would have a hope of following him.

Well, if there are no more questions I would be delighted to let you go around town.