# Stringy Topology Notes
# January 9, 2006

Gabriel C. Drummond-Cole

January 9, 2006

Please sign in on the signin sheet, so we can keep track of all the people who came. Another thing, Chinese people getting money from MSRI go and see the secretary, upstairs and to the right in the mathematics office. If you don't have your reimbursement forms, just ask for one. You'll mail those to MSRI. Enjoy your week and I'll see you around.

## 1 Uribe: Intro to topology

This is elementary to most of the people who are here. The idea is to give algebraic topology from scratch, then we'll do the Thom isomorphism, tomorrow $K$-theory, and then so on. I advise you to check out the downtown area. For the people who don't know, feel free to ask questions.

I'm going to give results without proofs.

We have here a topological space $X$ and I want to see how many holes it has, 1, 2, 3 dimensional holes, and how many subvarieties I can put in it up to perturbation.

I'm going to do singular homology. Take the $n$-simplex, the $n$-dimensional triangle $\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum t_i = 1, 0 \le t_i\}$. So $\Delta^0$ is a point, $\Delta^1$ a segment, and so on. Then I want to include vectors so that this is $\sum t_i \vec{v}_i$. Then look at a singular $n$-simplex, a map $\sigma : \Delta^n \to X$ I want to look at holes as built out of these maps. So let $C_n(X)$ be the free abelian group generated by singular $n$-simplices.

Let the boundary map $\delta_n : C_n(X) \to C_{n-1}(X)$ be $\sigma \mapsto \sum_{i=0}^{n}(-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$. Then $\delta^2 = 0$ and so I define $H_n(X) = \ker \delta_n / im \delta_{n-1}$.

The restriction map is the composition with the inclusion $\Delta^{n-1} \to \Delta^n$.

$H_*(pt)$ is easy to see to be concentrated in dimension 0. The $C_n$ are all generated by the unique maps $\Delta^n \to *$ so they are all $\mathbb{Z}$. So what happens with the boundary operator? If you have an even dimension, they cancel, and you get a zero map. If it's odd it's the identity. So

the complex is

$$\mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

Now if $f : X \to Y$ then $f_* : H_*(X) \to H_*(Y)$ and if $g \cong f$ we have $f_* = g_*$. Further $id_* = id$.

Now I need to discuss relative homology, homology for a pair. Let $i : A \to X$ be an inclusion. We define $C_n(X, A)$ to make this sequence exact:

$$0 \to C_*(A) \xrightarrow{i_*} C_*(X) \to C_*(X, A) \to 0.$$

This will give a long exact sequence in homology:

$$\ldots \to H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \ldots$$

These things in $C_*(X, A)$ are simplices in $X$ with boundaries in $A$. The new map $\partial$ is given by taking the boundary as a simplex in $A$.

We also have excision. Say $Z \subset A \subset X$. Then $H_n(X - Z, A - Z) \to H_n(X, A)$ s an isomorphism. There are restrictions on how $Z$ sits in $A$.

Now Mayer-Vietoris gives the homology of $A \cup B = X$. There is a long exact sequence

$$\to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to$$

This can be used to give the homology of the sphere $H_*(S^n) = \mathbb{Z}$ in dimensions 0 and $n$ and otherwise 0.

Take $B^n$ the $n$-dimensional ball. Then its boundary is $S^{n-1}$. Then $H_*(B^n, \delta B^n)$ is the same as $H_*(S^n)$ just from the long exact sequences.

So we want to build another gadget to calculate the same invariants which is easy to work with. We'll do this by building a space in stages, attaching balls to skeletons of lower dimension along the boundary.

$X^0$ is a set of points, the 0-skeleton. Between points we put paths, between paths 2-balls, and so on. So $X^k = X^{k+1} \sqcup B_\alpha^k /$. Take $\phi_a : S^{k-1} \to X^{k-1}$. Then $x \sim \phi_a(x)$ for $x \in \delta B_n$. Then we say $X = X^n$ for some $n$.

Then $C_n$ is the free abelian group generated by the $B_\alpha^n$. The boundary map $C_n(X) \to C_{n-1}(X)$ takes $e_\alpha^n \mapsto \deg \phi_a^n$. So $H_*(C_*^{CW}(X)) = H_*^{CW}(X)$ A proposition is that $H_*^{CW}(X) \cong H_*(X)$.

**Example 1** $\mathbb{RP}^n$ *is* $S^n/antipodes$. *So this can be built with one cell of each dimension, attaching with the two sheeted cover, meaning that the complex is*

$$0 \to \mathbb{Z} \to \ldots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

*So* $H_*(\mathbb{RP}^n)$ *is* $\mathbb{Z}$ *in dimension zero, and in* $n$ *if* $n$ *is odd, and* $\mathbb{Z}/2\mathbb{Z}$ *in odd dimension less than* $n$, 0 *otherwise.*

Now singular cohomology. Let $C^n(X) = Hom(C_n(X), \mathbb{Z})$. Then I get a boundary $\delta : C^n(X) \to C^{n+1}(X)$. This takes $F$, a map $C_n \to \mathbb{Z}$ to the map which takes $\sigma \mapsto F(\delta\sigma)$.

For $\mathbb{RP}^{\not{\,|\!\!\!\!\!\:E}}$ we get that $H^0 = H^3 = \mathbb{Z}, H^2 = \mathbb{Z}/2\mathbb{Z}, H_1 = 0$. So $H^n(X, \mathbb{Z}) \cong H_n/T_n \oplus T_{n-1}$, where $T$ is the torsion.

This is good because we have a ring structure given by the cup product which is $\phi \cup \psi(\sigma) = (\phi|_{\sigma[v_0,...,v_\ell]})(\psi_{\sigma[v_\ell,...,v_{\ell+k}]})$ for $\phi, \psi$ in $C^\ell(X, \mathbb{Z})$ and $C^k(X, \mathbb{Z})$.

Now some examples. $H^*(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\alpha]/\alpha^{n+1} = 0$. $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1} = 0$.

Let $M$ be a smooth manifold. An orientation is at every point a coordinate system so that I can move it around smoothly. I do this with $H^n(M, M - *)$. This is $H^n(B^n, \delta B^n) = \mathbb{Z}$. I say this is orientable if I can choose a generator of this group in a coherent way. So $x \mapsto \mu_x(H^n(M, M - x))$. I want to do this "smoothly on M" where this has some business with sheafs and stalks to actually define it.

What is not orientable? $\mathbb{RP}^{2n}$, the mobius band, and so on. What is an oriented space? One where I can choos this thing. If $\mu$ exists then it is oriented.

The cap product relates the homology and the cohomology.

$\cap : C_k(X) \times C^\ell(X) \to C_{k-\ell}(X)$. This takes $\sigma \cap \varphi$ to $\phi(\sigma|_{[v_0,...,\ell]})\sigma_{[\ell+1,...,k]}$.

This induces a cap produc $H_k(X) \times H^\ell(X) \to H_{k-\ell}(X)$. Now we can move on to Poincaré duality.

Let $M$ be an oriented compact $n$-manifold. What is a manifold? Take functions $\mathbb{R}^N \to \mathbb{R}$. Take the preimage of a regular point (derivative nonzero). Intersect these, making sure gradients are not dependent.

So suppose this manifold has fundamental class $[M]$. Then $D : H^k(M) \to H_{n-k}(M)$ by $\alpha \mapsto [M] \cap \alpha$ is an isomorphism. This is very nice.

Let's see a simple example. Look at $T^2 = S^1 \times S^1$. I am going to use deRham cohomology. So I have $d\theta, d\psi$. Then $[T^2] \cap d\theta = [S^1_\psi]$. Morally if you integrate along $d\theta$ you are left with the $\psi$ circle.

One last thnig. One can also think , if the manifold is smooth, the cup product can be thought of as intersection of varieties. Please correct me if I'm wrong. Suppose I have two manifolds. I have

$$
\begin{array}{ccc}
H^k \times H^{n-k} & \longrightarrow & H^n \\
\Big\uparrow{\scriptstyle PD} & & {\scriptstyle PD}\Big\uparrow \\
H_k \times H_{n-k} & \longrightarrow & H_0
\end{array}
$$

This is given by intersection, which should be transversal.

# 2    Lupercia, Transversality Thom Isomorphism

I am going to sketch some ideas. You should check Milnor's "Characteristic Classes" for more on this.

I will start by reminding you what is transversality. We start with two smooth manifolds $M^n$, $N^m$. This looks locally like $\mathbb{R}^n$ and varies smoothly as you move around it.

So start with $f : M \to N$ a smooth map. What does it mean for $y \pitchfork f$ for $y \in N$? Look at $d_* f : T_x M \to T_y N$ where $x$ is a preimage of $y$. Then you want it to be surjective. You can also call $y$ a regular value.

[Do you assume $n < m$?]

No, but that is a very good comment. If $m < n$ then it's the complement of the image.

This condition is important because what it's telling me is that the tangent space at $x$ translates to the tangent space at $y$ (?)

**Exercise 1**

*M is compact, then the regular values are open in N.*

**Exercise 2**

*If $m < n$ then $y$ a regular value means $f^{-1}(y) = \emptyset$.*

**Exercise 3**

*$y$ is regular means that $f^{-1}(y)$ is a smooth manifold of dimension $m - n$.*

**Exercise 4**

*Make sense of the saddle picture.*

**Theorem 1** *Brown-Sard*
*Let $W \subset \mathbb{R}^m$ be open and $f : W \to \mathbb{R}^n$ be smooth. Then the regular values are everywhere dense in $\mathbb{R}^n$.*

**Exercise 5** *Assume M has a countable basis. $f : M \to N$ is smooth implies the regular values are everywhere dense.*

Consider the second version of transversality. Let $Y \subset N$ be a submanifold of dimension $n - k$. Let $f : M \to N$ be as before. Then $f \pitchfork Y$ beans that $\forall x \in f^{-1}(Y)$, the composition $T_x M \to T_y N \to T_y N / T_y Y$ is surjective.

**Exercise 6** $f^{-1}(Y) \subset M$ is a smooth $m - k$ dimensional manifold.

I am going to give you the critical lemma, then state and prove that you can perturb either the map or the submanifold just a little and get transversality.

**Lemma 1** *Let $W \subset \mathbb{R}^m$ be open, and $f : W \to \mathbb{R}^k$ smooth. Let $X \subset W$ be relatively closed, and say $f|_X$ has 0 as a regular value, and that $K \subset X$ is compact. Then there exists $g : W \to \mathbb{R}^k$ smooth which coincides with $f$ outside a compact set, such that $g|_{X \cup K}$ has 0 as a regular value. And for all $\epsilon$, $|f(x) - g(x)| < \epsilon$ for all $x \in W$.*

itproof. Partitions of unity imply that there exists $\lambda : W \to [0,1]$ so that $\lambda = 1$ on a neighborhood of $K$ and 0 outside $K' \supset K$. Then $g(x) = f(x) - \lambda(x)y$

- 0 is a regular value along $K$, $d_x y = d_x f - \lambda 1(x)/y =_x d_x f$.

- $g(x) = g(k)$ on $W - K'$

- $|g(x) - f(x)| = |\lambda(x)||y| < \epsilon$

- $X \cap K'$ is compact, $|\lambda'|_{X \cap K'}| = X \cap K' \to [0,1]$ so $|\lambda'(x)y| < \epsilon'$ for small enough $y$ so $\partial g_i \partial x_j = \partial f_i / \partial x_j$.

**Exercise 7** $Trans(M, Y, N) \subset C^\infty(M, N)$ are dense. Find what topology.

Here is the third version.

$$M \xrightarrow{\ f\ } N$$
$$Y \xrightarrow{\quad j \quad}$$

Now $f \pitchfork j$ if $d_x f(T_x M) + d_y j(T_y Y) = T z N$ if $f(x) = j(y) = z$.

What are you doing? Knitting? My lecture is fascinating? This is the first time I have seen knitting. Eating sushi, but knitting?

## 2.1 Poincaré Duality

I am not going to give a more general definition, that is in every book, I think Harry Potter has it, but if $M^n$ is compact smooth oriented with no boundary then $H_i(M^n) \cong H^{m-i}(M)$. Take $r$ and $s$ and make them transversal. Suppose that $r + s = m$. Then generically $r$ and $s$ intersect at points. You can get a number using the orientation, that's the intersection number.

Then let me sketch the proof. Take a Morse function. By this I mean $f$ looks locally like $f(0) + \sum a_{ij} x_{ij} + O(|x|^3)$. Here $a_{ij} = 1/2 \partial f / (\partial x_i \partial X_j)$.

Remember Morse Theory. Then you have the saddle picture. You have $x$ the direction of positive eigenvalues, $y$ the direction of negative ones. These are the ascending and descending manifolds. Generically, I have two disks of complementary dimension for the positive and negative eigenvalues.

There is a beautiful argument to ensure that the value of the function at a critical point is the index (number of negative eigenvalues). Then between the points of index $i+1$ and $i$ you get disks. Then you take the intesection numbers of the disks in this picture from preimages of a point between them, and say $\delta e_i = \sum \langle \partial e_{i+1}, e_i \rangle e_i$.

Let $C_1 = \{\sigma :$critical points of index $1 \to \mathbb{Z}$. Take the free abelian group on these.

**Exercise 8** $\delta^2 = 0$.

$\langle e_{i+1}, e_i \rangle = \langle \tilde{e}_{i+1}, \partial \tilde{e}_i \rangle$. There are more algebraic proofs using homological algebra.

The spirit of these lectures will be to use smooth manifolds.

## 2.2 Thom Isomorphism

Suspensions.

Well, $\tilde{H}_i(X) = H_i(X, *)$.

**Proposition 1** $\tilde{H}_i(\Sigma A) = \tilde{H}_{i-1}(A)$

$A \times [0,1]/A \times \{1\} = CA \cong *$. Then $\Sigma A = CA/A \times \{0\} = CA/A$.

So what do you do? You consider the pair $(CA, A)$ and take the sequence

$$\to H_i(CA) \to H_i(CA, A) \to H_{i-1}(A) \to H_{i-1}(C, A) \to$$

And so you get an isomorphism between the middle two of these, which eventually gives what you want.

You can suspend multiple times.

**Exercise 9** *Compute $H_i(S^n)$. Note that $\Sigma^n S^k = S^{k+n}$.*

A vector bundle associates to every point of a space a vector space, with some local properties. You can get the suspension from a particular equivalence on a trivial bundle, $A \times \mathbb{R}^n / \sim = \Sigma^n A$.

If I do the fiberwise one-point compactification I will get a sphere bundle $Sph(E)$. Now I take the disk bundle $D(E)$, the vectors of norm at most one in the vector space. There is also the sphere bundle $S(E)$ of norm one.

**Definition 1** $T(\xi) = D(E)/S(E)$.

Notice that there is a map $\pi : Sph(E) \to T(\xi)$ by identifying points at infinity. Then the Thom diagonal Takes $Sph \to Sph \times Sph \to A \times T(\xi)$.

Then I do identifications and I get a map $T\xi \to A_+ \wedge H\xi$, the Thom diagonal.

In cohomology it looks like $H^p(A, R) \otimes \tilde{H}^q(T\xi, R) \to \tilde{H}^{p+q}(T\xi, R)$.

**Exercise 10** *If the bundle xi is trivial then $\mu \in \tilde{H}^q(T\xi, R)$ where $\mu$ is the suspension of the identity.*

*Then $H^p(A; R)$ is equivalent to $H^{p+q}(T\xi, R)$. This is just the suspension isomorphism when this is a trivial bundle.*

We're almost done.

**Definition 2** *An R-orientation for xi is a $\mu \in \tilde{H}^n(T\xi, R)$ such that for all $a \in A$, $i_a^*(\mu) \in \tilde{H}^n(S_a^n)$ for $i_a : S_a^n \to T\xi$.*

**Exercise 11** *Apply the Serre spectral sequence to the fibration $S^n \to Sph(E)$. Show $\exists \mu$ implies this collapses, so $H^*(Sph(E), R) cong H^*(A, R) \otimes H^*(S^n, R)$.*

**Exercise 12** *If there exists a section of $\infty$ then $H^*(A, R) \cong H^*(A, R) \otimes H^0(S^n, R)$.*

**Exercise 13** *Prove the Thom isomorphism theorem.*

# 3  Uribe: Semisimplicial spaces

Please correct me if I'm using the wrong names. What am I going to talk about is a combinatorial gadget that captures and organizes topological information of categories, groups and some infinite dimensional spaces.

They will be constructed using simplices. You can build a circle with one point, by taking an interval and a point, and identifying the boundary of the interval, both sides, with the point. You can put two two simplices onto a circle with three points to make a sphere.

I am just joining along the boundary to get something of a higher dimension.

Say I have a finite group $G$. I want $BG$ with $\pi_1(BG) = 0$ and $\pi_i(BG) = 0$ for $i > 1$.

So for each element of $G$ I add a loop to a point. To create the relations, inside any relation I put a cell. I get something eventually in the limit which is $BG$.

So $BG = EG/G$ where $EG \cong *$. $G$ acts freely on $G$. $B\mathbb{Z}_2 = S^\infty/\mathbb{Z}_2 = \mathbb{RP}^\infty$, and $BS^1 = S^\infty/S^1 = \mathbb{CP}^\infty$.

Now let $\mathscr{C}$ be a category. A semisimplicial object is a contravariant functor $Ord \to \mathscr{C}$. where the objects of $Ord$ are the natural numbers and $Mor(Ord)$ are the order-preserving maps.

This gadget $n$ is like the simplex $\Delta^n$. $n \to m$ are generated by the maps as follows

- $\delta_i : (n-1) \to n$ which is injective and doesn't contain $i$. These are the face maps of a simplex.

- $\sigma_i : n+1 \to n$, the unique surjective map which hits $i$ twice. These are the degeneracy maps.

You could alternately look at $\Lambda : Ord \to \mathscr{C}$. There are three arrows going down from $A_2$ to $A_1$ and two going up. It's easiest to do it as a contravariant functor.

So for me now, $\mathscr{C}$ will be the category either of sets or of topological spaces. They will be called semisimplicial sets or spaces. I can associate to it a geometric realization from this data. So $A_*$ yields the topological space $|A_*|$.

This realization is $\sqcup \Delta^n \times A_n / \sim$ . The equivalence relation, if $\theta : n \to m$ then $\theta_* : \Delta^n \to \Delta^m$. We say $(x, A(\theta)y) \sim (\theta_*(x), y)$.

Now let $\mathscr{C}$ be a small category, with the objects and the morphisms topological spaces. Then we can associate the nerve of the category $N\mathscr{C}$. This is a semisimplicial thing.

$N\mathscr{C}_0$ are the objects, $N\mathscr{C}_1$ are morphisms, $N\mathscr{C}_n$ are the composable $n$-tuples of morphisms.

So my face maps are $\sigma_1 \cdots \sigma_n \mapsto \sigma_1 \cdots (\sigma_i \circ \sigma_{i+1} \cdots \sigma_n$ and the degeneracy maps are to introduce copies of the identity morphism.

So we get $\mathscr{C} \to N\mathscr{C}_* \to |N\mathscr{C}_*| = B\mathscr{C}$. By looking at what happens with a simple composition, we see that nothing more than what we wanted happens.

Let $G$ be a discrete group. Then there is a category which I will also call $G$. There is one object and the morphisms are $G$. The nerve $NG_n$ is $G^n$. The realization is then $BG$.

So $F : \mathscr{C} \to \mathscr{D}$ can be checked to induce a continuous map $BF$ between classifying spaces. If $F$ and $H$ are related by a natural transformation of functors then $BF$ and $BH$ are homotopic.

$Ad_g : G \to G$ is equivalent to $id$. Conjugating is homotopically equivalent to the identity.

One can construc some other categories from a finite group. You can define the following, $\bar{G}$ has $G$ as objects and $G \times G$ as morphisms. Then $(g, h)$ is an arrow from $g$ to $h$. This maps to $G$ by sending all objects to the single object of $G$. and $(g, h)$ to $gh^{-1}$. So $\bar{G} \sim id$ so $B\bar{G} \cong * \to BG$. This is like $EG \to BG$, but you have to work a little bit.

So $A_*$ a semisimplicial space. $Top$ goes to chain complexes by taking $X$ to $C_*(X, R)$. Then

gor $A_n \to A_{n-1}$ you get $C(A_n, R)$ and $C(A_{n-1}, R)$. Then $C_*(A_n) \to C_*(A_{n-1})$ takes $\sigma$ to $\sum_0^n (-1)^i CA(\delta_i)\sigma$.

The homology of this complex is the Hochschild homology $HH(A_*, R)$.

**Theorem 2** *Burghelea*
$HH(A_*, R) \cong H_*(|A_*|, R)$.

[Why did we do this?]
To compute homology.

I think Dennis will use this. For simplicial spaces (sets) there was one thing that always troubles me. The arrows go in the wrong direction. I always thought that it should be "co." Why not cosimplicial? If that's a simplicial space, what is a cosimplicial space? It's a covariant functor $Z : \Delta \to \mathscr{C}$. These also have a geometric realization, it's a little more complicated.

$|Z^*|$ is the space of infinite tuples in $\prod Map(\Delta_k, Z_k)$ such that the following commutes:

$$
\begin{array}{ccc}
\Delta_k & \xrightarrow{\ f_k\ } & Z_k \\
\uparrow{\scriptstyle \delta_i} & & \uparrow{\scriptstyle Z(\delta_i)} \\
\Delta_{k-1} & \xrightarrow[\ f_{k-1}\ ]{} & Z_{k-1}
\end{array}
$$

**Theorem 3** *(Jones, Siegel)*
$HH_(C^*(Z_*)) \cong H^*(|Z_*|)$.

Now I want to construct the loop space. Construct the following semisimplicial space. $\lambda^n(m)$ are morphisms from $m$ to $n$. This is a simplicial space. The face and degeneracy maps are the natural ones.

**Exercise 14** $|\lambda^n_*| \cong \Delta^n$

Why do all of this? Look at $\lambda^1(m)$. There are $m + 2$ of these. So what is happening with the faces and degeneracies? Let $X^{\lambda^1}$ be the cosimplicial space such that $X^{\lambda^1}(n)$ is $Map(\lambda^1(n), X)$. At the level $n$ this is basically $X^{n+2}$. So you get $\delta_i(x_0, \ldots, x_{n+1}) = (x_0, \ldots, x_i, x_i, \ldots, x_{n+1})$ and $\sigma_i(x_0, \ldots, x_{n+1}) = (x_0, \ldots, \hat{x}_i, \ldots, x_{n+1})$. So the first and last points are invariant here, like the beginning and endpoints of the path.

Now $|X^{\lambda^1}_*| \cong X^{|\lambda^1|}$ which is just the path space. If I just take the subcosimplicial space where the first and last are the same, everything goes through as before, so you get $X^S_* \subset X^{\lambda^1}_*$ with $|X^S_*| \cong \mathscr{L}X$.

This may illuminate the theorem that $HH(C^*(X^S_*)) \cong H^*(\mathscr{L}X)$. You have $X, X^2, X^3$ and you associate to these $C^*(X)^{\otimes 1}, C^*(X)^{\otimes 2}, C^*(X)^{\otimes 3}$. Dualizing you get $Hom(C^*(X)^{\otimes j}, C^*(X))$.

If $\pi_1 = 0$ then $HH(C^*(X^S), C^*(X)) \cong H_*(\mathscr{L}X)$. I don't have time to say that Poincaré duality gives you the product of Chas and Sullivan.

# 4    Discussion

I have $X^*$ a cosimplicial space and then I get $C^*(X^*)$ But I can also look for $X^S$, because of the way it's constructed, you can construct from $C^*(X)$ the Hochshild homology of $X$. The correct statement is $H(X^*) \cong HH(C^*(X))$. Hochschild homology is of an algebra, not a cosimplicial space.

Let's discuss. I want to discuss something. How do you prove that $|\lambda^n| \cong \Delta^n$. The only object in the construction that survives is the identity map. Let $k > n$. I believe $\sigma \in \lambda^n(k)$ then ...

[Basically you can recover homology through this, can you get $K$-theory simplicially?]

You have $[X, \mathbb{Z} \times BV]$, here you have a simplicial model for $\Omega V$.

You can do it with derived categories. The category of categories (a principle of Grothiendieck) is nothing but a subcategory of the simplicial category of sets.

Let me say something that I forgot. This is a way to make a simplicial space for a manifold using an open cover. $M_{i-1} = \sqcup_{j_1,\dots j_i} \cap^i U_{j_i}$.

If things intersect you join them. It's like gluing. This gives you something homotopically equivalent to $M$.

If you form a category $\mathscr{M}$ with morphisms which are triples $x, i, j$ with $x \in U_i \cap U_j$ and the objects are pairs $(x, i)$ with $x \in U_i$. So $U_1 = \sqcup U_{i,j}$ and $U_0 = \sqcup U_i$. This is essentially the information you need for a vector bundle.

A functor $g : \mathscr{M} \to VECT$ associates to $(x, i)$ a vector space $E_{x,i}$ and to $(x, i, j)$ the isomorphism $T_{x,i,j}$. You have a vector space over every point but there is nothing ensuring it varies continuously. One way to resolve this is to take a smaller category with only one vector space in each dimension. You can also do something fancy, with say Fredholm operators or something else you might like better because of your particular perversions.

Felix, what didn't you understand of my lecture? You weren't there?

[Take the functors from $\mathscr{C}$ to itself given by the identity and by taking everything to the final object.] So the identity and the constant map on $B\mathscr{C}$ are homotopic.]

**Exercise 15** *Consider the category $\mathscr{J}$ which has two objects $0$ and $1$ and one nonidentical morphism $0 \to 1$. Then $F \cong G$ as functors $\mathscr{C} \to \mathscr{B}$ if $H : \mathscr{C} \times \mathscr{J}$ with $H|\mathscr{C} \times 0 = F$ and $H|\mathscr{C} \times 1 = G$.*

*$\mathscr{C} \in CAT$, with natural transformations $F \to G$. There is a $2$-functor to $Top$ with continuous*

*maps and homotopies.*

The gadget $X^S$ is a cyclic space, it has an action of the circle. $\tau_i : X_n^S \to X_n^S$ with $(\tau_i)^{n+1} = id$. The cyclic homology $HC_*(X^S)$ is isomorphic to $H_{S^1}^*(\mathscr{L}X)$, equivariant cohomology of the loop space $H^*(\mathscr{L}X \times ES^1)$.

[Tangent on a complex no one can remember precisely.]

So $\Lambda \to Top$ where $\Lambda$ is a cyclic category. You want to map the points on the circle to the points on the circle preserving the cyclic order now instead of an order.

[What about Quillen's proof of transversality?]

I would butcher it. It is very ingenious, vory tight, it's in Differential Topology, Guillemin and Polluck.

The way I proved (or did not prove) the Thom isomorphism, you don't need the Serre spectral sequence. The proof in Milnor and Stasheff uses only the classical properties, excision and Mayer-Vietoris. Most proofs copy the proof of Milnor. That's a book you should read to do this kind of stuff. String topology uses topology but the nice thing about it is that a lot of it can be done with basic first year topology. A nice thing is that things about this come from physics, most naturally the quantum structure of the world. If you have a space you can associate to it an algebra. You have a space and a functor associates an algebra to its space. So say $X \to C(X)$. These could be the complex valued continuous functions (or smooth or something. You can add them, multiply by members of the ground field and multiply. This is called an algebra. This is surprisingly fruitful. You can recover the space by studying the algebra. This is surprisingly fruitful at least to me.

In physics we don't just have a space, we have a physical system. You take a manifold plus a structure, that can help something. So maybe a metric. Then this gives an algebra of some sort. They have some additional structure. These can be Lie or Poisson or Gerstenhaber or BV. What is most remarkable about string topology. Classical topology gives you the same algebras as physics! The exclamation point is to connote our extreme surprise and liking of this, this is a connection between two seemingly disparate things, classical algebraic topology and the algebras of physics.

Today we torture you a bit with classical topology so you can appreciate the connection when it comes. That's the plan for the week.

[Can you give us a concrete example for how to calculate the cohomology of the loop space by the Hochschild complex?]

Luc?

$HH_*(C^*(X), C^*(X)) \cong H_*(\mathscr{L}X)$. So $C^*(X)$ is a differential graded algebra, very big. In many cases it will be formal, your space, so you can have a map $C^*(X) \to H^*(X)$ which induces an isomorphism. YOu compute it on $H^*$ instead. Now take the sphere. Take $X = S^m$, so we have the homology of the sphere is an exterior algebra on an element of

degree $n$, $x_n$. If you have $A$ an augmented algebra, you have a complex $\mathscr{C}_*(A, A)$ which will be $A \otimes T(\Sigma \bar{A})$. Then $\mathscr{C}_*(H^k(S^n), H^k(S^n)) = E(x_n) \otimes T(x_{n-1})$. In this case modulo two the differential is zero. The differential in general, with odd sign, is $a \otimes [sa_1 \otimes \cdots \otimes sa_k] = aa_1 \otimes \cdots \otimes a_k \pm \sum a \otimes [a_1 \otimes \cdots a_i a_{i+1} \otimes \cdots a_k] \pm a_k a_1 [a_2 \otimes \cdots \otimes a_{k-1}]$. All the products in the exterior algebra are zero. You will get two or zero, $(1 \pm 1) x \otimes x^{k-1}$. So the final result is that $H_*(\mathscr{L}X, \mathbb{Z}/2\mathbb{Z}) \cong E x_n \otimes T(X_{n-1})$, where this is $H^k(S^n) \otimes H^k(\Omega S^n)$.

Thank you very much, we continue tomorrow.