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As a mathematician you can get respect and pay for going very deeply into your own world. As a corollary of that, if someone asks you to help organize a workshop in honor of someone, you want to. Then listening to all of the talks, you realize that this choice, of hyperbolic geometry, you could have chosen any number of other subjects, it's a really special thing.

## 1

I'm going to talk about 2-generator hyperbolic three-manifolds. Some of you who were at Cameron Gordon's conference might notice this is the same topic. I do have new results but I may not be able to talk about that today.

**Definition 1** *The rank of a group is the minimal number of generators.*

How do we describe a 3-manifold such that the rank of the fundamental group is less than or equal to $g$. Is there some topological prescription for producing these manifolds. The analogous question for one and two dimensions is completely solved; in four or higher, any group arises so possibly this is uninteresting. In three dimensions this may be a tractable question.

Today I will be considering the compact case, but if one considers noncompact manifolds with finitely generated fundamental group, then there's a theorem of Scott and Shalen that says the group is finitely presented, and, moreover, $M$ is homotopy equivalent to a compact three manifold. So to understand this question we can reduce to the compact case.

**Definition 2** *Heegard splitting of a compact manifold is a decomposition into two compression bodies.*

In this case those can be handlebodies. So we say $M = H_1 \cup_{\delta H_1 \sim \delta H_2} H_2$. The rank of $\pi_1(M) \leq h(M)$, the Heegard genus for $M$ closed. The fundamental group of either handlebody is a free group, so it can generate the group, with relations from the other.

In the general case, with compression bodies, if one side is a handlebody, then this inequality still holds.

This is the only general method known for constructing three-manifolds with a restricted rank. A natural question is, how much can rank $\pi_1(M)$ and $h(M)$ differ, in the closed case? What I conjecture is that

**Conjecture 1** *There exists some function* $\mathbb{N} \to \mathbb{N}$ *such that* $h(M) \leq F(rk\pi_1(M))$.

Boileau-Zieschang had that there exists infinitely many closed Seifert fibered spaces such that the rank is the Heegard genus minus one. Then Shultens-Weidmann showed that there exist graph manifolds $M_i$ such that $h(M_i) - rk(\pi_1 M_i) \geq i$. In this case $h(M_i)$ is always at least a constant times $i$. There are no known examples of hyperbolic three-manifolds for which the rank and Heegard genus differ. Very little is known in general for hyperbolic manifolds.

**Theorem 1** *Adams*
*If $M$ is a hyperbolic manifold of finite volume with $\pi_1(M)$ generated by two parabolic elements, then $M$ is a two-bridge link complement.*

A simple example is the figure eight knot complement, whose fundamental group is generated by the two parabolic elements $\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \omega & 1 \end{bmatrix} \rangle$ with $\omega^3 = 1$.

When I glue two things together by a four braid, I get two generators and then one relator. It's also not hard to see that the rank, which is two, it's also equal to the Heegard genus of $M$. You can see a decomposition into a handlebody and a compression body right here.

I want to sketch why this is true.

The idea of the proof is that we take this two-bridge link and put an order $\pi$ orbifold locus on the link. So I get an orbifold, I drew a two-bridge link, but imagine there are two parabolic generators. The orbifold is generated by taking the meridian and specifying that it has order two. Then $\pi_1(\mathscr{O})$ is a dihedral group. Then by the orbifold theorem, you have to check that this is irreducible, is a spherical orbifold, and its twofold branched cover is a cyclic group. It will be a spherical orbifold with cyclic fundamental group so a lens space. There's an elliptic involution on the torus that extends over the solid torus. If I quotient by the involution, I get a two-sphere with two $\pi$-orbifold loci. There's a little geometry you have to do but it's basically the argument. This argument goes back to Boileau-Zimmerman. If it were an infinite cyclic group you find it's the trivial knot or link, or you might get a torus knot, there are a bunch of exceptional cases I'm suppressing.

I also know about torus bundles, which come from taking $T \times I$ and identifying the top and bottom by an element of the mapping class group. So $\varphi_* = A \in SL_2\mathbb{Z}$. Let $M$ be the mapping torus of $\varphi$.

**Lemma 1** *$M$ is two-generator if and only if $A$ is conjugate to* $\begin{bmatrix} m & 1 \\ -1 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$. *Then the rank of $\pi_1(M)$ is $h(M)$*

You can notice here that $m$ is arbitrary so the trace can be arbitrary. In $SL_2\mathbb{Z}$, well, in $SL_2\mathbb{Q}$ everything is conjugate to this form, this is rational canonical form [??? How can that be, you have trace and determinant related by D=T+1], but this is not everything over $\mathbb{Z}$

Take this [graphical] element of the mapping class group, so a little neighborhood of this gives a genus two handlebody. Then we take this, and we can generate our torus as well as the mapping class element.

The way to show that if it's not one of these, then the rank is at least three is, check [unintelligible]. It should send one generator to anether so that these two guys would generate the fundamental group of the torus. If I take a guy with a fixed point you take a neighborhood and get a punctured torus bundle that is hyperbolic and the rank is the same as by Dehn filling on the boundary, and also the Heegard genus.

So this is also true for punctured $T^2$ bundles.

These are generally Solv, but there are other Solvs which are the union of two twisted $I$-bundles over a Klein bottle. This is probably a feasible case to consider.

The theorem I want to talk about in the remaining time is

**Theorem 2** *(Agol) Given $\epsilon > 0$ there exist at most finitely many closed hyperbolic three-manifolds with rank $\pi_1(M) = 2$, $inj(M) > \epsilon$, $h(M) > 2$.*

If you fix rank two and a small epsilon, there are infinitely many with Heegard genus greater than two. This is due to Namazi. It depends on some unpublished work of Tien.

These are created by taking genus two handlebodies and a map $\varphi$, and raise it to a high enough power, a pseudo-Anosov element, and the geometry splits into two pieces looking like Shockey groups, and the ending laminations, it starts to look almost like a fibered thing fibered over the circle with monodromy $\varphi$.

The result of Tien is that you can get arbitrarily close to being hyperbolic as you glue together. You need to be close in some integral sense, the trace of the Ricci curvature is close to zero or something like that.

**Theorem 3** *(Souto)*
*Given $\epsilon > 0$ there exists a $g$ such that if $M$ is hyperbolic, rank $\pi_1 M = 3$ and $inj(M) > \epsilon$, then $h(M) \leq g$.*

If $M$ is infinite volume and rank $\pi_1 M = 2$ then we know $\pi_1 M$ is free so $M$ is the interior of a handlebody (Jaco-Shalen plus tameness), which simplified the rank two case.

An application:

**Conjecture 2** *Short geodesic conjecture*
*There exists a lower bound on the injectivity radius of arithmetic hyperbolic three-manifolds.*

I don't have time to go into the definition of arithmetic; in the noncompact case they're all commensurable with [unintelligible], like $PSL_2(\mathscr{O}_d)$ where $\mathscr{O}_d$ is a ring of integers in $\mathbb{Q}[\sqrt{d}]$. The short geodesic conjecture is a special case of Lehmer's conjecture

**Conjecture 3** *If the monic polynomial $p(x)$ has integral coefficients then $M(p) = 1$ or $M(p) \geq 1.7\ldots$*

The measure $M(p)$ is the product of the absolute value of the roots lying outside the unit circle. This known lower bound is the Mahler measure of $x^10+x^9-x^7-x^6-x^5-x^4-x^3+x+1$.

A corollary of these two theorems is, assuming the short geodesic conjecture, there exists only finitely many arithmetic hyperbolic $M$ with rank of $\pi_1(M) \leq 3$. This is a corollary of Souto's results in the rank three case, but uses the rank two. This uses Gelbart-Jacquet, Jacquet-Langlands, and Vigneras. The theorems they prove translate into lower bounds on [unintelligible]. It's also based on a technique of Lackenby, to get an upper bound on the Cheeger constant which is comparable to the first eigenvalue.

Now I just wanted to explain, I do have a generalization in the rank two case where I don't have the injectivity radius but I need other technical assumptions. Now I want to sketch the argument of the rank two case.

Step one is to find a short generating set for $\pi_1 M$. This works if you take it mod $\epsilon$ which is the Margulis constant. Then there exists a rank two graph $\Gamma$ and a map of $\Gamma$ into $M$ such that $\pi_1(\Gamma)$ maps onto $\pi_1(M)$ and $M$ has a thick-thin decomposition, so the thin regions are going to be Margulis tubes, solid tori, and the length of $\Gamma$ intersected with the thick part is uniformly bounded, less than some constant which doesn't depend on the manifold. There are a few options, but one of the generators will be very short.

The sketch of the the proof is, take a pair of generators, and if $\gamma_2$ has short translation length then we can take a Margulis tube around it. If this and $\gamma_1$ are too far apart then they would just generate a Shockey group. I could find a geodesic plane to bisect one from the other. This plane will be taken to a disjoint geodesic plane and so on, so you can play ping-pong if you like, but I want to take $H^3/\Gamma_2$ and then I'm taking an infinite volume solid torus, take a melon-baller and scoop it out, and glue them together. It's another way of seeing a combination theorem. The axes can't be too far apart. If they're close together and one is very long, you multiply the one by some power of the other, and shorten the carrier graph.

Now you have the carrier graph, with bounded length. If the injectivity radius is bounded below, then we can find loops arbitrarily far away. Now I drill three loops very far away. You can think of drilling them out and then putting in a pinched negatively curved metric, or do a trick of Canary where these are orbifold loci.

Then you take a cover corresponding to the image of $\pi_1(\Gamma)$ which says the image of $\{\pi_1\Gamma\}$ is a free group.

Then tameness, Gabai talked about this on Monday, this branched cover is tame. This is $\pi_1\mathscr{O}' = \pi_1\Gamma$. The idea is , we've taken this nasty branched cover, it will be far away from the carrier graph. It will be geometrically finite and have bounded area, a genus two surface with bounded area. There's a method to interpolate this through a sequence of bounded surfaces until we get one that hits our carrier graph. When we project these back we get an immersed thing of genus two that separates these. You promote the immersed surface to an embedded one, and then you have a few more steps, it uses the techniques used for proving tameness.

[If you have a three manifold, you can talk about the algebraic rank, and also the geometric rank, the number of handles you need for a decomposition. If you take a four-manifold and cross with an interval is the geometric rank of the four-manifold the same as the algebraic rank?]

[Let's thank Ian once more and meet again at 10:35]

# 2 Souto, volume of hyperbolic three-manifolds and distance in the pants complex

[It's a pleasure to introduce Juan Souto.]

Thank you very much, happy birthday, so, mmm, I will start, $M$ is a three manifold, orientable, closed. Thurston's hyperbolization conjecture asserts if $M$ is irreducible (that means $S^2$ in $M$ bound balls) and if $\pi_1(M)$ is of finite order and $\mathbb{Z}^2 \nsubseteq$
$pi_1(M)$ then $M$ is hyperbolic, a quotient of $\mathbb{H}^3$ by $\Gamma \subset PSL_2\mathbb{C}$ discrete and torsion free.

parts of this were done by Thurston, Perelman, and:

**Theorem 4** *(Mostow)*
*If $M$ admits a hyperbolic metric then this metric is unique up to isometry.*

Okay, so now it comes a little bit of political propaganda, the geometric invariants are then topological invariants: volume, $\lambda_i(M)$, and so on.

But given a topological description of $M$ what can be said about the metric?

In Perelman's approach one can hope that one starts with a metric that looks like being hyperbolic, but one must be careful, but suppose, well, for example for all $L$ and $\epsilon > 0$ there exists $M$ with two metrics $g$ hyperbolic and $g'$ not hyperbolic, but $||K_{g'} + 1||_\infty < \epsilon, ||K_{g'} + 1||_{10} < \epsilon$ but $(M, g)$ and $(M, g')$ are not $L - bi - Lipschitz$.

Political propaganda again, allows you to reduce the one kind of argument to another, the hyperbolization conjecture.

I want to consider $M$ which are, consider $M = H_1 \cup H_2$ a Heegard splitting into handlebodies. I want to say this is strongly irreducible, meaning that each $D_1 \subset H_1$ and $D_2 \subset H_2$ (properly embedded and essential, which means removing them hurts a handle) then $|\delta D_1 \cap \delta D_2| \geq 2$.

**Theorem 5** *For all $g$ there exists $L_g$ so that for all $M$ and a strongly irreducibl Heegard splitting of genus $g$ $L_g^{-1}\delta(H_1 \cup H_2) \leq vol(M) \leq L_g \delta(H_1 \cup H_2)$.*

This $\delta$, there are many choices for it, we'll choose it for the theorem to hold. So let $\Sigma = \delta H_i$ be the Heegard surface, and let $\mathscr{P}(\Sigma)$ be the pants complex, a graph with vertices pants decompositions, and edges elementary moves. These are replacing a curve in the pants decomposition with another that intersects it minimally. So $\mathscr{P}(\Sigma)$ is connected (Hatcher Thurston) so this is what one uses to measure distance.

So $\Sigma$ bounds a handlebody to the left and to the right, which gives me two pants decompositions. So $\delta(H_1, H_2) = \min\{d_{\mathscr{P}(\Sigma)}(P_1, P_2) | p_i \in \mathscr{H}(H_i)\}$. Now $\mathscr{H}(H_i) = \{P \in \mathscr{P}(\Sigma) | \exists D_1, \ldots, D_{g-1}$, with $\delta D_j \in P$ and $H_i \setminus \cup D_k$ is [unintelligible]of solid tori$\}$. One would think that you just want every circle to bound a disk, but without a bound on the lower injectivity radius you need these solid tori.

The upper bound is easy. Essentially, given a path of length $\ell$ in the pants complex joining the two handlebody sets, a path of length, one obtains, there is an ideal triangulation of $M$ with six or four, what is it, $6\ell$ simplices. You consider the two handlebodies the same and have the gluing map. [unintelligible][unintelligible]. And then the volume of $M$ is $v_3||M|| \leq 6\delta H_1 \cup H_2)$.

How does one get the lower bound? Somehow, it's funny, in the notes of Thurston there's a picture showing that something is a Heegard splitting of a three manifold and that's the last time this appears.

I'm also going to assume for simplicity that for all disks the boundaries of the pairs of curves bind the surface, that is, there are no other curves disjoint to these two.

**Theorem 6** *(Pitts-[unintelligible])*
*$\Sigma$ is isotopic to a minimal surface*

Associated to a Heegard surface you have a sweepout. I get a family $S_t$, for $t$ in the middle this is $\Sigma$ and $t$ at the end it is a graph. Given such a sweepout, one takes maximal area, and then minimize over all sweepouts, then you get a sequence of surfaces. Onee has to be a little bit careful, one has to modify the choice of the surface. The surfaces $S^m$ will converge to a minimal surface as measures (as [unintelligible]).

The fact that this convergence is this way, one doesn't know about what this converges to, but one has to, it's, well, the theorem holds.

So now you have $M$ divided by a minimal surface $\Sigma$. There is a theorem of Alexander-[unintelligible]that the two pieces, $\tilde{U_1}$ is a CAT$(-1)$ space. So both handlebodies $U_1$ and $U_2$ behave almost like Sholtley groups.

So $P_\Sigma$ is the shortes pants decoposition on $\Sigma$. Then $vol(U_1) \geq L_g^{-1} d_{\mathscr{P}(\Sigma)}(P_\Sigma, \mathscr{H}(H_1))$.

Now if the injectivity radius is bounded below, we have $vol([S_1, \Sigma]) \geq 1$ and $d_{\mathscr{P}}(P_1, P_\Sigma) \leq L_g$. If the length of the compressible curves in $\delta U$ is sufficiently large then I find $S_1$.

[At this point I stopped taking notes.]

# 3 Shalen, Hyperbolic geometry and classical topology

I was reminded of one thing by Ron Stern saying he'd heard of John long before he met him. I heard of him, Dennis Sullivan was giving lectures on a bunch of stuff, the work wasn't written up, they were trying to pin him down on details, Dennis said, "There's a guy named Morgan, talk to him, he knows how all these things are defined and fit together."

We collaborated on hundreds of pages, and after a suitable cooling off period we get along again.

It's scary, really, the way you've kept alert, paid attention to all the talks.

[It's what you heard about when you're a graduate student, you just pay attention to one phrase, like "what about the non-simply connected case"]

This question is again getting an explicit understanding of Mostow rigidity. So the topology should determine the geometry, but what are the explicit connections between topological and geometric invariants? They were addressing qualitative questions and we're trying to address quantitative questions.

It's an old result, I don't know whom to attribute it to, if you put an uppor bound on the volume of $M^3$ hyperbolic then this puts an upper bound on the Heegaard genus nad this gives an upper bound on the rank of $\pi_1$ and tus the rank of $H_1(M, \mathbb{Z}_p)$. The bounds you get are absolutely terrible, the question is whether you can do anything more realistic.

The work is based on a theorem. part was proved by Anderson, Canary, Culler, S., but we could only prove a weak version of it.

We needed the tameness conjecture, Agol; Calegari-Gabai. So the theorem I have in mind, I'll call it today

**Theorem 7** *The strong* $\log(2k-1)$ *theorem*
*Suppose you have $k$ elements $x_i$ in $PSL_2(\mathbb{C})$ and they freely generate a subgroup which is discrete. Also assume that there are no parabolics, I don't know why I'm putting this in.*

*Then for any $p \in \mathbb{H}^3$, we have*

$$\sum_1^k \frac{1}{1+\exp(d_i)} \leq \frac{1}{2}$$

*where $d_i$ is the hyperbolic distance from $p$ to its image under $x_i$, $d(p, x_i p)$.*

It is assumed that the group is free here, because of this condition in order to apply it, you need serious topological information about the manifold. Sometimes you can address this by classical methods. That will be the general theme of the talk.

I want to say a few words about the proof of this, I won't give a sketch of the proof but I want to say something about the underlying philosophy. There are what is called Patterson-Sullivan measures, Patterson considered them for Fuchsian groups and then Sullivan very extensively for Kleinian groups. Let $\Gamma$ be a Kleinian group, and then they constructed a measure on the limit set $\Lambda(\Gamma) \subset S^2$. It's got some nice properties I'll say in a bit; it's constructed as a weak limit of measures supported on an orbit in $\mathbb{H}^3$. Each measure in the sequence is a sum over certain quantities on the orbit, so it's over the group, which indexes the orbit.

What Culler and I noticed, I really should have mentioned the corollary

**Corollary 1** *For some $i$ it follows that $d_i \geq \log(2k-1)$.*

We'd proven this for $k = 2$ under other hypotheses (not having the tameness conjecture).

In the case when $\Gamma$ is free, we'd noticed, well, let's call it $F$, it can have a decomposition, the same one as in the Banach-Tarski paradox. You have $F_{x_1} \cup F_{\bar{x}_1} \cup \ldots \cup F_{\bar{x}_k} \cup \{1\}$ where these are the words beginning with these letters. So this decomposes the Patterson Sullivan measure as a sum $\mu = \mu_{x_1} + \ldots + \mu_{\bar{x}_k}$.

If I multiply $F_{x_1}$ on the left by $\bar{x}_1$ then I get $F - F_{\bar{x}_1}$. Basically what this says about $\mu$ is that if you take the pullback of $\mu_{x_1}$ by $\bar{x}_1$ you get $\mu - \mu_{\bar{x}_1}$. This also satisfies a transformation law similar to the one for area. It can be interpreted, I won't write down formulas, of a power, the critical [unintelligible]of the conformal expansion, what Sullivan calls this.

The integral is a little mysterious, but the conformal expansion factor is explicitly defined as a function of $d_1$. So maybe by using some of these identities you get some information about the size of $d_i$. If you look at the worst case, things work great. There the limit set is the entire sphere at infinity, and now moving forward to the present, with the tameness theorem, you get the strong ergodicity properties for the action of the group on the limit set, so the mystery measure $\mu$ is nothing but the ordinary area measure, so the whole thing reduces to a calculus problem and you can solve it.

This is not a sketch but a philosophical outline, I think, of the proof.

You can use this to study the geometry of hyperbolic manifolds, let me give you an example, my way of thinking about Margulis tubes. Suppose you take, to be concrete, $M^3$ a closed orientable hyperbolic 3-manifold, think of it as $\bar{H}^3/\Gamma$ where $\Gamma \cong \pi_1 M$, which we will assume to have no rank two subgroup of finite index. Agol mentioned a theorem of Jaco and me, also Tom Tucker (unpublished) which says that every rank two subgroup of $\Gamma$ is free. You consider in this case the log 3 displacement cylinders. Say $X$ is a maximal cyclic subgroup of $\pi_1(M) = \Gamma$, then the log 3 displacement $Z_X$ are points $p \in \mathbb{H}^3$ such that $d(p, xp) < \log\ 3$ for some $x \neq 1$ in $X$. The theorem says this is a cylinder

What follows from the log three theorem is that if you take two distinct maximal subgroups, these cylinders $Z_X$ and $Z_{X'}$ are disjoint. Otherwise you would [fast argument]

If these are disjoint then they cannot cover hyperbolic space, since space is connected. If you take a point not in this union its image in the quotient is the center of a ball of radius (log 3)/2. Then using sphere-packing estimates you can conclude, you can get a lower bound for the volume of $M$, getting at least .92. If you work much harder, you can get, this was Culler-Horsovsky-Shalen, and then Przeworski improved this, using Gabai-[unintelligible]-Thurston. I think Andrew's bound was bigger than one, 1.0... I said I was going to relate volume to topology. There's a topological argument, Shalen-Wagreich, if $\pi_1 M$ has a finite index subgroup of rank two, then $H_1$ with mod $p$ coefficients has rank at most three. In general that argument goes $r$ to $r+1$. If the volume, in particular, is less than one, then you can say $H_1(M, \mathbb{Z}_p)$ has rank at most three.

I want to make comments on this statement. One is that by working much much harder one can strengthen this result. Mark and I gave a Dehn surgery argument, using a result due to Agol-Dunfield, which in turn uses Perelman's estimates, and what we proved was the same statement, we also used Przeworski's tube packing estimates, we got the same bound on homology using only the assumption that the volume of $M$ is less than 1.219, that the rank of $H_1$ is less than or equal to three for mod $p$ coefficients. I think there's very classical kinds of conjectures about Heegaard genus which will play a major role.

**Conjecture 4** *For any closed hyperbolic three-manifold $M$, the Heegaard genus of $M$ is equal to the rank of $\pi_1 M$.*

**Conjecture 5** *This I formulated at a conference last summer, the experts thought it was interesting and plausible. If $M^3$ closed and hyperbolic, orientable, and the Heegaard genus is $g$, then any finite sheeted cover of $M$ has Heegaard genus at least $g-1$. It's known that it can go down, even under a two sheeted cover.*

The previous conjecture would imply this for two-sheeted cover, but this would be for more than two.

Jointly these would imply the Heegard genus version of what I wrote down a minute ago, if the volume of $M$ is less than 1.219 then the Heegaard genus of $M$ would be less than or equal to three.

I hope I remember to point out that these would imply Heegaard genus versions of something else I'm talking about in a little while.

Those are some instances of what you can do under the assumption that every two-generator subgroup is free.

Another thing that was in the paper with Anderson, Canary, and Culler, is the following result: if every subgroup of rank at most three in $\pi_1 M$ is free, then $\mathbb{H}^3$ cannot be covered by log 5 displacement cylinders defined by maximal cyclic subgroups of $\Gamma$. This is the same definition as before. From this it follows that $M$ contains a ball of this diameter. This gives a much stronger lower bound on the volume, 3.08. This is much much harder to prove than the corresponding two-free case. You might think naively that three of them would meet in a point and then the three corresponding things would generate a cyclic subgroup, but that might be of rank two, not rank three. You have to study the nerve of a covering, a two-complex, and combine topological properties of the nerve with group theoretical information about the lattice of free subgroups, there's a quite involved combinatorial argument. Well, that statement brings up an obvious problem, when can you say a subgroup of rank up to three is free? Say that $H_1(M, \mathbb{Z}_p)$ has rank at least five for some $p$.

[How big is the largest disk in the thrice punctured sphere?] I wrote down five because that's what I need to show that every three-generator subgroup has infinite index. You can't assume that it's free. Take the compact core of some cover, and that shows that it's a handlebody of genus two, or it has an incompressible boundary of genus less than two, so then it's free. But in the three case you can have an incompressible surface of genus two.

So in this case it's either 3-free or $\pi_1(M)$ contains a genus two surface group. In the paper we wrote about this, the paper I'm about to talk about is on the arXiv, it's going to appear in Transactions, I forget what it's called. This statement is not in the paper, we only realized it in this sense later thanks to Gabai, it's a topological theorem,

**Theorem 8** *Suppose that $M^3$ is closed, hyperbolic, orientable, and $\pi_1(M)$ contains a genus two surface group. Then either $H_1(M; \mathbb{Z}_2)$ has rank at most ten or $M$ contains an incompressible surface of genus two or three. This is a book of $I$-bundles with [unintelligible].*

I can only say a few words about the proof, which is really classical topology with a vengeance. This plus very deep results of Agol-Storm-Thurston implies

**Theorem 9** *Suppose $M^3$ is hyperbolic, closed, orientable, and suppose that the volume is at most* 3.08. *Then $H_1(M; \mathbb{Z}_2)$ has rank at most ten.*

I'll indicate why this follows, and we can definitely improve ten to seven. Six seems to be much harder than seven. There are some very fascinating questions about finite two-groups. I won't say anything lower than ten in the talk. Let me indicate how you would prove that. For the proof, the mere fact that $H_1$, assume the rank is at least eleven. Then, since it's at least five, either the volume is greater than 3.08 or $\pi_1(M)$ contains a genus two surface group. Then the topological theorem says it's rank at most ten, which we've assumed away, or it contains an incompressible surface of genus either two or three. What Agol-Storm-Thurston showed, in particular, is that either the volume of $M$ is greater than 3.66, which is the volume of a regular ideal hyperbolic octahedron, or $F$ is a "fibroid." This is a generalization of being a fiber over $S^1$. This means you look at $M_F$ which is $M$ split along $F$. To say that $F$ is a fibroid means that each component of $M_F$ is a book of $I$-bundles. Take a polyhedron in a three manifold made up of simply closed curves and surfaces of negative Euler characteristics attached along boundary by covering maps.

Using not much more than Mayer-Vietoris, you can show that if the, if, actually, the topological theorem should say, genus two or separating of genus three. So if it's a fibroid with the stated restrictions on genus, then you have a bound on the rank of $H_1$, maybe $\pi_1$ which is $\leq 7$.

I only have a couple of minutes to chat about the topological theorem. Apart from the business of [unintelligible]it's in the same region as Dan's lemma. You can realize the surface group in the group, and then homotope it into good position, pass to a covering, and keep going. Classically it's not hard to show this terminates, but a little harder for us to put things in good position. So $H_1F$ maps onto $H_1M$ so upstairs there isn't much homology. Then it splits into cases. In one case you use Poincaré duality and in the other Gabai's theorem about immersed surfaces being representable by embedded ones.

Then [fast argument] and go down the tower, so that every stage you pass to a book of $I$-bundles instead of an incompressible surface.

Sometimes this doesn't work because you want to modify by cutting and pasting but the manifold is a closed manifold mapped onto by the book of $I$-bundles. The machinery I described earlier says that if you started out with a lot of homology then a sheeted cover will have lots of homology, so you have a contradiction.

This is a vast simplification. You can sharpen it to seven but it's much more technical to get down to six.