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1 Eliashberg, DEs of Symplectic Field Theory

When I was in Russia, I knew Morgan as just a name on the covers of books, like Milnor, Poincaré. My title involves differential equations. I'm not really an expert in integrable systems. I'm thinking about this in the hopes that there's an expert in this area that will explain what I'm talking about.

The appearance of integrable systems in relation to GW theory is nothing new, but I think the symplectic field theory part is new.

$\mathbb{CP}^2$ has n points, we want to count how many rational curves d pass through these n points, call this $N_{n,d}$. Then you can make a function $f(t,z) = \sum N_{n,d} t^n z^d$. I will explain how this function appears as a solution to a DE in symplectic field theory. Let's do the following procedure. I'm just talking here about genus zero.

So consider the space E of, formal loop space in $\mathbb{C}^2$, so two functions, $U(x) = (u_0(x), u_2(x))$ for $x \in S^1$. These are formal, I don't discuss any convergence. I assume $\int_0^{2\pi} u_j(x)dx = 0$. So I think $u_0 = \sum p_{k0} e^{ikx} + q_{k0} e^{-ikx}$. I want to think about this as a symplectic space, with the p, q canonical coordinates. I think of the inner product with matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then the symplectic form is $\sum_{k=0}^{\infty} \frac{1}{k} dP_{k\gamma} \wedge dq_{k\bar{\gamma}}$ for $\gamma = 0, 2$. So we have the bracket $\{dU, dV\} = \frac{1}{2\pi}\int_0^{2\pi} \langle dU, dV' \rangle dx$. Consider Hamiltonian flow $H(U) = \frac{1}{2\pi} int_0^{2\pi} (\frac{u_0^2}{2} ? e^{u_2 - ix}) dx$. Then $u_0 = i(e^{u_2 - ix})_x$ and $u_2 = i(u_0)_x$. Then $\ddot{u}_2 = -(e^{u_2 - ix})_{xx}$.

Consider flow and take a certain Lagrangian submanifold, the zero section. Flow the zero section to time t and you get L_t, a Lagrangian submanifold. In general this might not be graphical but in formal power series everything is graphical. I get a function $f_t(q)$ defined by $p_{k2} = k \frac{\partial f_t}{\partial q_{k0}}$ and so on. Take a $q_{1\,0} = z$ and all other q equal to zero. Consider the value at this point. This value will be some function $\varphi(t, z)$, and that will be the function mentioned before.

The goal of my lecture is to give a hint of an explanation of this.

I could do a change of notation, instead of t I write t_4 and instead of z I write ze^{t_2}. Then I add a constant term $\frac{t_0^2 t_4}{2}$. These correspond to the different constraints, t_0 no constraint, the t_2 divisor constraint, and the other one [unintelligible]. Now I could repeat what I said before, except I would plug in in $q_{1\,0} = ze^{t_2}$. In the Hamiltonian Jacobi equations the constant is determined, and the constant I have given is correct. Now you can say, do E^3. Then you have like (u_0, u_2, u_4). The inner product matrix is like $\begin{bmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{bmatrix}$. So you now look at

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \varphi}{\partial t_4}(t_0 + u_0, u_2, u_4, e^{-ix}) dx = H$$

Then you play the same game as before and you get the answer for $\mathbb{CP}^3$. So the answer for the point was $t_0^3/6$ which becomes $t_0^2/2$ by differentiating, then you add the u_0 and get $\frac{1}{2\pi} \int_0^{2\pi} \frac{(t_0+u_0)^2}{2} dx$.

Let me start with some of the explanation for higher genus. You consider the space of meromorphic functions on Riemann surfaces. So for S_g you consider a divisor $D = (d_1, \ldots, d_n)$ of total degree zero. Then $J_g^D \subset \mathscr{M}_{g,|D|}$ is the space of Riemann surfaces with divisors such that D is rationally equivalent to zero. We have the canonical $\mathbb{C}^* = S^1$ bundle. The meromorphic function will be determined uniquely by a complex number, but I'll be ignoring the $\mathbb{R}$ component so it's S^1.

Now what you observe is the following, the compactification of the total space $\bar{\tilde{J}}_{g,n}^D$. The normal way you compactify, the more nodes, the higher the codimension. What I want to think of is that the meromorphic functions splits, the poles in one direction, the zeros in the other. They have matching conditions in their poles and zeroes. The codimension is still one, you cannot [unintelligible]any node. It's codimension one, no matter how many nodes. If I fix a meromorphic function, I have a canonical map to $\mathbb{C}$. I have to match them. You might have a rotation. I can only multiply the whole meromorphic function, not separately at each pole.

So you get some, if you denote by the union of all these $\bar{J}$. There is an operation of gluing on these, if I denote this by $\circ$, so $\delta J = J \circ J$. This may be similar to what Dennis will say. He thinks about the moduli space itself. Here you have some kind of different algebra.

Now you have the marked points. I can consider the evaluation map. If you take $J_{g,n}^D$. I can associate ψ classes, the degrees of the Chern class of the canonical line bundle over the marked point. Then you can pull back the homology of S^1, get $\psi_i^k \cdot 1$ and $\psi_j^k \cdot \theta$. Then you take this moduli space and integrate these classes and get an analogue of Gromov Witten potential, getting H. If I introduce t_k for the ψ crasses and τ variables so $\sum t_k \psi^k \cdot 1 + \sum \tau_k \psi^k \cdot \theta$. Then $H(t, \tau)$, I say a pole of multiplicity k is p_k and a zero is q_k. So $H(t, \tau, q, p, \bar{h})$.

So a pole of mulitplicity four and zeroes of multiplicity $2, 2, 1$ is $p_4 q_2^2 q_1$. So the integral of this thing over the moduli space is zero. The algebra of gluing is the same as the algebra of differential operators. So think q_k, p_k generate an algebra with the relation $[q_k, p_k =]k\bar{h}$. This gives $p_k = k\bar{h}\frac{\partial}{\partial i_k}$. So operators are $\sum q_i q[p]$, [unintelligible], $D_1 \oplus D_2$.

You have this function H and all variables are graded. The t and τ are graded by the homology classes they represented. All my moduli space are odd dimensional because I quotient by $\mathbb{R}$. So H is odd but the only odd variable is τ. Hence, if I write the expansion I get $\sum \tau_i H_i + o(\tau)$. So if you write $H \circ H = 0$ this means $\sum \tau_i \tau_j [H_i, H_j]$. So you get this a sign of a quantum integrable systems. An infinite system of commuting differential operators.

[The higher terms correspond to an L_∞ version?]

I have no idea. I don't think anybody considers such problems in algebraic geometry.

So now, what does it have to do with what I was explaining at the beginning? Suppose now I have a Riemann surface with some punctures, and now I consider the GW-invariant of this one,$\mathbb{C}$. So I consider maps of Riemann surfaces into it, I'm trying to get relative GW-invariants. So then I can write down the same way as before a generating function for this,

$f(S, p)$, where S is similar to τ. There are no zeros, only poles. If you take $\frac{\partial e^f}{\partial s_i} = \frac{\partial H}{\partial \tau_i}|_{\tau_i=0} e^f$. So start with a solution without constraint and solve an equation of evolution. As always with this kind of quantum formalism, this specializes to classical mechanics and you instead of the commuting q, p I should get something whose Poisson bracket is k and instead of [unintelligible]I get Hamilton Jacobi which is what I get before. That was the semi-classical version of this.

Now if, why did I do this computation of the generating function? I wanted something for a closed manifold. What if I did this for $\mathbb{CP}^1$? That's why I need a gluing formula.

So, what is an SFT? It's a functor from a category $Geom_{SFT}$ to Alg_{SFT}. So in $Geom$ there are odd dimensional manifolds with extra structure and symplectic cobordisms. Then on the other side I get (W, H), a Weyl algebra and an element H with $H \circ H = 0$. This has coefficients in a polynomial ring. This has a representation as differential algebra of a "Fock" space. Then because $H^2 = 0$ we can take homology. If you have a cobordism you have two ends and with every end you associate its own Weyl algebra, so you get W_+ and W_- You have q_+ and q_- variables and p_+, p_-. So you have $f(q_-, p_+)$ Then this thing is like, well, p_+ is like a fifferentiall operator, apply this to $\varphi(q_+)$ and then evaluate at $q_+ = 0$. No this gives a function $\hat{\varphi}(q_-)$ which commutes with H and gives a chain map.

This is just one strategy. There is a gluing formula that comes from [unintelligible]. Another important part is the evolution equation coming from something. The variable t come from a cohomology class or a differential form. If you have Θ on the manifold, then $\theta = \Theta|_{\delta W}$ Then $F(\theta)$ and $H(\theta)$ define evolution equations.

I think I should stop.

[Can you write the evolution equation?]

$\Theta \rightsquigarrow T, \theta \rightsquigarrow t$. So

$$\frac{\partial e^{F_\theta(T,q)}}{\partial T} = \hat{H} e^{F(T,q)}$$

where

$$H_\theta(t) \frac{\partial H}{\partial T}|_{t=0} = \hat{H}.$$

2 Khovanov

[Note. This talk was functionally the same as the Khovanov talk given two weeks earlier]

I need to start by talking about bimodules and Hochschild homology. So fix an algebra R over a field k and we have M an R-bimodule. Depict this as M in a box with lines from the top and bottom signifying the left and right action. You can tensor bimodules by sticking them together pictorially.

$$\begin{array}{c} {\scriptstyle A}\,\big| \\ M \\ {\scriptstyle A}\,\big| \\ N \\ {\scriptstyle A}\,\big| \end{array}$$

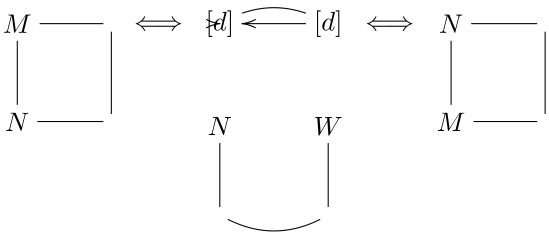
R just as a bimodule over itself is just a line. since $R \otimes_R M = M$.

So eventually this will be a derived tensor product. If M is a right R-module and N is a left R-module then the usually tensor product $M \otimes N$ is an Abelian group, but the derived tensor product exists, $M \hat{\otimes}_R N$ is: take a projective resolution P_i of bimodules projective on the right of M, tensor every term with N, and at this point you can either take homology or just think in the derived category. Later they will all be projective on the left and on the right, so we won't have to do this.

Now we can close things off, attach the lines coming out of the top and bottom and connecting them to one another by looking at $M/[M, R]$, symmetrizing by setting the left and right actions equal to one another. These are the R-coinvariants of M. The quotient is only a vector space. The quotient functor is right exact and subobject functors are left exact, so we can do another derived functor. We can take a resolution of A by projectives. That's because $M_R = M \otimes_{R \otimes R^{op}} R$

So we take the derived tensor product so we choose a resolution of M, or better, R, and then for any bimodule we can do the same thing. we convert R into a complex of projective $R \otimes R^{op}$-modules, and you get what we call the Hochschild homology of M; $HH(M) = H(M \otimes_{R \otimes R^{op}} \tilde{R})$.

Any module has the biresolution. To be thrifty we can take a smaller resolution. Let me note that this interpretation of the closure respects the composition of the bimodules. That is, $HH(M\dot{\otimes}N) \cong HH(N\dot{\otimes}M)$. There's the mysterious picture in the middle where it becomes W.



So say $r = \mathbb{Q}[x]$. Then this is $\nvdash \to Q[x] \otimes \mathbb{Q}[x] \to \mathbb{Q}[x] \otimes \mathbb{Q}[x] \to \mathbb{Q}[x] \to 0$. The differential here first takes $1 \otimes 1$ to $1 \otimes x - x \otimes 1$, and the next one is multiplication.

When you tensor you get $0 \to M \to M \to 0$ with the map $m \mapsto mx - xm$. So $HH_0(M) = M_R$ and $HH_1(M) = M^R$, the coinvariants and invariants. This is a coincidence. Usually this would not be so nice, M^R is defined as $HH^0(M)$ but this will not always be $HH_1(M)$.

What about for $R = \mathbb{Q}[x_1, \ldots, x_n]$? We can tensor together the last resolutions. We can take $R \otimes R \to R \otimes R \to 0$ where $1 \otimes 1$ goes to $x_i \otimes 1 - 1 \otimes x_i$, and tensor n copies of this together. So we get 2^n copies of $R \otimes R$ in a complex. So for HH we need to tensor with M giving 2^n copies of M taking $m \to x_i m - m x_i$. In this case it will be $HH_0(M) = M_R$, the coinvariants, and all the way up to $HH_n(M) = M^R$ the invariants.

Now let $R_i \subset R$ be the subring of polynomials invariant under the permutiation of x_i and x_{i+1}. As an R_i-module, this is free of rank 2, with $R = R_i \cdot 1 \oplus R_i x_i$.

We can write $R = H^*_G(G/B)$ and then $R_i = H^*_G(G/P_i)$ where $G = GL(n, \mathbb{C})$ and B is Borel.

If R and M are graded then $HH(M)$ is bigraded. So take $B_i = R \otimes_{R_i} R$, this is projective as a left and as a right R-module. We'll keep track of grading, giving each x_i degree two. I'll explain why two and not one later. So for instance $B_i \otimes B_i$ is $R \otimes_{R_i} R \otimes_{R_i} R = B_i \oplus B_i\{2\}$. So this is $2 \cdot 2 = 2 + 2$.

There is a bimodule map $B_i \to R$ taking $a \otimes b$ to ab. This is a complex of bimodules, $0 \to B_i \to R \to 0$. There's another like $0 \to R \to B \to 0$ which takes 1 to $(x_i - x_{i+1}) \otimes 1 + 1 \otimes 1 \otimes (x_i - x_{i+1})$. Just shift the degree of B_i down by two to make the complex differential have degree zero.

Take the simplest braids σ_i in the braid group Br_n; this is generated by σ_i which interchanges i and $i+1$. Assign the two complexes I've just generated to $\sigma_i^{\pm 1}$. So what is $F(\sigma_i) \otimes F(\sigma_i^{-1})$? This is $B_i \otimes R \to B_i \otimes B_i\{-2\} \oplus R \to R \otimes B\{-2\}$. I wanted to choose the grading of my modules so that R is always in degree zero.

Notice that everything is over R so we get a simplification:

$$0 \to B_i \to B_i\{-2\} \oplus B_i \oplus R \to B_i\{-2\} \to 0$$

There is a differential which we didn't write down. If you want to write down what ∂ is, you find out it decomposes into $0 \to B_i \to^1 \to B_i \to 0$,
$0 \to B_i\{-2\} \to^1 B_i\{-2\} \to 0$ and $0 \to R \to 0$. In the homotopy category, complexes of graded R-bimodules $F(\sigma_i) \otimes F(\sigma_i^{-1}) \cong R$. So tensoring two complexes of the generating braid and its inverse together yields the complex of the trivial braid.

Theorem 1 *$F(\sigma_i)$ gives rise to a braid group action on this (homotopy) category $\mathscr{C}$. This means $g \to F(g)$ and $F(gh) \cong F(g)F(h)$ and $F(1) = Id$. Eventually we want natural equivalences.*

We want $F(\sigma_i) \otimes F(\sigma_j) \cong F(\sigma_j) \otimes F(\sigma_i)$ trivially, and $F(\sigma_i) \otimes F(\sigma_{i+1}) \otimes F(\sigma_i) \cong F(\sigma_{i+1}) \otimes F(\sigma_i) \otimes F(\sigma_{i+1})$.

We've gotten to braids, but we want to get to links. To do this we take the closure of a braid. So $\sigma \mapsto \hat{\sigma}$. So closure, remember the beginning of the lecture, should correspond to taking the Hochschild homology. So starting with σ we take $HH_F(\sigma)$. This has $\to F^j(\sigma) \to F^{j+1}(\sigma) \to$ where F_j is a direct sum of tensor products af B_i. This is additionally graded and the differential preserves the grading. If you start with a graded ring, the Hochschild homology is bigraded instead of being merely graded.

I will get a bigraded vector space $HH(F^j(\sigma)) \to HH(F^{j+1}(\sigma)) \to$, with each term bigraded. The differential preserves the bigrading. We get a complex of bigraded vector spaces. So now we can take homology again since $HH(\partial)^2 = 0$ since HH is functorial. So $H(\sigma) = H(HH(F(\sigma)), HH(\sigma))$.

Why do this?

Theorem 2 *$H(\sigma)$ is triply graded, depends only on $\hat{\sigma}$ and has Euler characteristic equal to the HOMFLY polynomial of links. This is a modification of Rozansky, Khovanov*

The HOMFLY is uniquely determined by the conditions $\lambda P(L_+) - \lambda P(L_-) - (q - q^{-1})P(L_0) = 0$ and the value of $P(\text{unknot})$. For instance, the polynomial of the two component unlink is $\frac{\lambda - \lambda^{-1}}{q - q^{-1}}$. This should be expanded as a power series.

For example, σ is the trivial braid. Then $F(\sigma)$ is R and $H(L) = HH(R) = R \otimes \wedge(y_1, \ldots, y_n)$.

You can get a braid group action for any Weyl group but it will only pass to links for this case.

Ideally you want this to be a functor. You want, for a completely legitimate link homology from link cobordisms to some algebraic category. The objects will be oriented links and then morphisms isotopy classes of surfaces with these links as boundary in $\mathbb{R}^3 \times I$. If you restrict to trivial cobordisms between trivial links, then you can get a two dimensional TQFT so we would need the H of the unknot to be a Frobenius algebra. It's a commutative associative algebra and coalgebra with a unit and a counit. Any Frobenius algebra is finite dimensional. But $H(\text{unknot})$ is $\mathbb{Q}[x] \otimes \wedge(y)$ which is infinite dimensional. But we want to reduce this to a finite dimensional thing by getting rid of this, that is the goal.

I still have five minutes so let me say in conclusion, going back to bimodules, you encounter arbitrary tensor products of B_i. But this decomposes into $\bigoplus_{w \in S_n} B_w^{n_w}$, and the interpretation is $B_w = H_G^*(IC(\bar{\mathscr{O}}_W))$. Here $\mathscr{O}_W \subset G/B \times G/B$.

[You were saying link homology should be a functor, what if it's concordances?]

How do you prove it? You need to use the braid representation of the link so it's impossible to say anything about cobordisms.