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1 Kirby

Since I'm the oldest speaker you have to cut me some slack. There are people here who got their PhD before me, they were younger and smarter. I have to leave today, so I'll miss the banquet when we're supposed to tease John Morgan, so I want to do that right now. I can only think, it was probably 1970. [Anecdote]

I've actually thought of John as having the most lucid mathematical mind I've known. He's played a major role in all of the topology of the last forty years. He started with surgery. Papers with Sullivan, and then along came the Thurston hyperbolic geometry revolution, when Thurston announced his theorem about hyperbolization of sufficiently large three manifolds. It was lots of notes and lectures, not really written down, it was said that Sullivan and Morgan were the first ones who understood it. Then he got into gauge theory. That lasted until Seiberg Witten in 1994. Then there was a fling with mathematical physics, and then, of course, as you know, the book on Perelman is supposed to be ready any time now. What did he miss in the last forty years? He's been in the middle of basically all the large events. And we're all waiting to see what comes next.

I wanted to talk today on some work I've been doing with David Gay. You can find notes on this on his website. He's at Cape Town in South Africa. You can find notes there and you don't have to take any now. So this concerns four-manifolds. The Lefschetz fibrations, for example a K_3 surface. This is roughly equivalent to the four manifold being symplectic. Certainly there are lots of nonsymplectic four manifolds.

So recently we got this theorem fo Aureau-Donaldsons-Katz[unintelligible], where every near symplectic manifold has a singular Lefschetzz pencil. If $b_2 > 0$ then X has a near-symplectic form, i.e., ω such that $d\omega = 0$ and $\omega \wedge \omega$ is a volume form, but ω vanishes on circles. The model for this is, look at a Morse function $f : M^3 \to \mathbb{R}$. In a neighborhood of a critical point, on $N \times S^1, df \wedge d\theta + \star df$, or something like that, and you get a whole circle where the two-form vanishes. That's what a near-symplectic form is, and now, what's a singular Lefschetz pencil?

The pencil part says you have to blow up a finite number of times. I'm not going to say much about that. The singular part means we have this projection to $\mathbb{CP}^1 = S^2$. On the left side we have a pencil and on the right side we have a pencil, but the fibers have different genuses. So what happens over the equator is we add round one-handles to raise genus across from one side to the other. A round one-handle is a circle cross an ordinary one-handle. This is exactly like the model I described.

This is what we were given and this is about two years ago. To somebody like me, I want to construct these things. The theorem we have, it's similar but a little bit different.

Theorem 1 *X is smooth oriented closed. Then X is a singular achiral Lefschetz fibration.*

There are four points of difference. They have pencil, so they have to blow up, we don't but we have achiral, which is a defect. They have to have positive Betti number so they can't deal with homotopy two spheres.

Let me go back and say, if you look at the circle bundle around a nodal fiber, it's given by a diffeomorphism of the fiber, made up of a right-handed Dehn twist. There's a handedness there, so these things have right (only) handed Dehn twists. We allow left handed Dehn twists. That's a defect and we wish it wasn't there. They also have all of their round one-handles, which can be added independently of one another. We don't know if ours can be added independently. We also have some round two-handles. You might have to lower the genus and raise it later on.

So it's a different theorem. So what's the argument for proving this? If you think of the typical Lefchetz fibration, you can imagine building the two halves and then gluing them together. You're trying to glue along a surface bundle over a circle.

You have to fit them together with a diffeomorphism, that's too hard for us. Instead you want to imagine, instead of drawing the two-sphere this way I want to draw a plane so I can draw the equator like this. We want to take out of one of them a disk cross a disk. You find a little plug, $e^2 \times e^2$, and add it to the other side. That has the advantage that the object that's left has a bounded fiber. The boundaries are open books. There's a binding and then all those pages. The boundary is the binding, and the pages glue in to the binding at the top and bottom, well, along the circle.

By putting the plug in the other side we can now glue along open books. We have [unintelligible]'s theorem, the one piece of advanced technology. This says two open books are stably equivalent if the associated two-plane [unintelligible]are the same. The construction, you can make the two-plane fields agree and then stabilize and glue together and get the achiral Lefschetz fibration.

Let me do a simpler case, with some homotopy four-spheres. We have in the back of our minds is that, maybe there's exotic ones or maybe they're all the same. What is this kind of homotopy four sphere. There's an old, in the 1960s, Fox said there are bushel-baskets of possible counterexamples of Poincaré, some branched coverings over knots. After these were resolved, there were no counterexamples proposed.

There are several bushel baskets af possible counterexamples for the smooth four-dimensional Poincaré. Take a knotted S^2 in S^4. Its normal bundle is $S^1 \times S^2$. Cut it out and glue in with the only interesting diffeomorphism, where you rotate S^2 as you run around S^1. This is only known to be a four-sphere in some few cases.

The ones I'm using, suppose you take $\langle x, y | xyx^{-1} = yxy^{-1}, x^4 = y^5 \rangle$. This is the trivial group, since $xy^4x^{-1} = yx^4y$ and eventually get $x^4 = y^4$. But suppose you build something up, a homotopy 5-ball using this relation, gluing handles for these things. Is it the five ball? Nobody knows how to do this. You want to do handle slides that mirror the algebra. When you take the fourth power, how do you do that by geometry? I have to have another two handle I can slide over it two times. To do that I need an extra handle to slide over this one, but I don't have an extra handle. One alternative is to add a $(2, 3)$ cancelling pair. With the extra 2-handle I can do that, maybe two cancelling pairs. At the end of that you'd still have extra three handles, and that problem is no better than the first.

These are interesting candidates. If one could simplify this presentation, then one would be done. "Can you trivialize this without remembering?"

Another way to see this boundary, you build a homotopy four-ball. And then cross with I. If you do it that way you see that the boundary is the four ball, doubled. We're looking at a class of homotopy four spheres that are doubled homotopy four balls. There are a lot of different choices, but when you cross with I they are the same. Our construction is, we have the one handles, and those are paired off. There are two handles attached along the one-handles. That's the kind of picture you have for the homotopy four-ball. Each two-handle has a double, we have this picture, now we can arrange whatever framing we want by sliding over the zero sphere. The purpose of this is, you can find a fibration of the four-ball so that the, there exists a Lefschetz fibration of the zero handle and the one-handles so that the two handles with negative framing lie on the pages with framings equal to $TB - 1$. These are vanishing cycles. That's because we got the framing right. We were able to do this because we made them all negative. Those handles are then nice vanishing cycles. They're right-handed Dehn twists because they're the good kind of cycles.

The zero handles we slide over their companion two-handles. When you do that, after you do the slide, this matrix operation shows that it links the original two-handle, and it lies in the same page, because anything that lies in the same page has framing TB. So because of this matrix entry we have a left handed twist.

The virtue to this is, the Lefschetz fibration we have, we changed the handlebody structure by writing a zero handle as a smaller zero handle and a lot of one and two handles. We have a higher dimensional fiber cross a disk and then we have pairs of cancelling cycles. We built something with bounded fiber, we have not capped off the fiber, and in the boundary the monodromy is trivial, first a right and then the left handed Dehn twists. This is then just, along the boundaries, a connect sum of $S^1 \times S^2$. Out here we have 0 and 1-handles. Just briefly, the zero handle will become $S^1 \times B^3 \cup B^2 \times B^2$. This two handle will become the plug that fills in and the other one fiber singularly ovr the 2-disk. It has a closed fiber which is S^2. There are some tricks in the one-handles. In the other construction they drop to the two sphere, and then add back up with one-handles. It doesn't require work to glue together because there are no interesting diffeomorphisms of connect sums of $S^1 \times S^2$, and it's even easier because we have trivial monodromy.

2 Carlson

Let me say a few words about cubic varieties. There will be overlap with things I've said elsewhere. Everything is joint work withh Allcock, Toledo. There is some previously unsuspected differential geometric stuff about some algebraic valrieties. Let me begin with elliptic curves. They're double covers of the Riemann sphere, or you can think of them as cubic hypersurfaces. You can take the geometric invariant theory point of view, you can think of cubic forms modding out by projective linear transformations. This is the GIT moduli space. YOu can also think of it as the upper half plane modulo a nice group $\mathscr{H}/PSL(2,\mathbb{Z})$. You use a period map. This goes back to Picard, maybe further. The upper half plane has curvature -1 so this is a hyperbolic orbifold.

Let's recall as a warmup how this goes. If you have an elliptic curve $\mathscr{E}$ then you pick a basis for homology, and then you have the holomorphic one form $\omega = dx/y$. Picard's construction is, you associate to an elliptic curve the number τ which is the ratio $\int_\gamma \omega / \int_\delta \omega \in \mathscr{H}/Gamma$.

In modern parlance say $m : H^1(\mathscr{E},\mathbb{Z}) \to \mathbb{Z}^2$. Then $\mathscr{E} \mapsto [m(H^{1,0}) \subset \mathbb{C}] \in grass(1,2) = \mathbb{P}^1$.

We would want to get some better control over the metric in a more general situation.

Let me tell you what happens in the case of cubic surfaces. The moduli space is a four dimensional space. We want a period domain of dimension four and an isomorphism. It's also something with six points blown up. Look at $m : H^2_0(S,\mathbb{Z}) \to \mathbb{Z}^6$ and then we look at the Hodge decomposition, but all of it is type $(1,1)$ so nothing comes out of this. This is probably why this theorem wasn't proved in 1975. There's a trick due to Picard. The surface $S \subset \mathbb{P}^3$ means that you can construct a three to one cover of $\mathbb{P}^3$ branched along this surface. You say $F(x_0,x_1,x_2,x_3) + x_4^3 = 0$. Then $H^3(T) = H^{2,1} \oplus H^{1,2}$. No you can look to $H^{2,1} \subset \mathbb{C}^10$ which is in $grass(5,10)$ of dimension 25, which is too high, and then something kills ten of those dimensions, and now you have deck transformations.

The first piece of structure is $H^3(T) = H^3_\omega(T) \oplus H^3_{\bar{\omega}}$ and then the latter of these breaks up into $H^3_{\bar{\omega}} = H^{2,1}_{\bar{\omega}} \oplus H^{1,2}_{\bar{\omega}}$ where the first of these is dimension one, the second of dimension four. Then $\Phi = \frac{dx_1 \wedge dx_2 \wedge dx_3}{f(x_1,x_2,x_3)^{4/3}}$ and then $[mH^{2,1}_{\bar{\omega}} \subset \mathbb{C}^5] \in grass(1,5)$ and then there are a couple more things. I have to talk a little bit more about the marking. At this point a little bit of what a geometer would think of as arithmetic. $\mathscr{E} = \mathbb{Z}[\omega]$ (which is a third root of unity). Then $H^3(T,\mathbb{Z})$ is a rank five $\mathscr{E}$ module. You can multiply ω by a class c giving $\sigma^* c$. This module has a hermitian form on it, from the cup product and so on. This can be diagonalized to $(-1,1,1,\ldots,1)$. Then the marking will just be $H^3(T,Z) \to \mathscr{E}^{1,4}$. Then to compute a period vector, find a basis for $H^3(T)$ as this kind of module, and define $mH^{2,1}_{\bar{\omega}}$ as $\mathbb{C}(\int_{\gamma_i} \Phi)$. And you find out that $-|z_0|^2 + |z_1|^2 + \ldots + |z_4|^2 < 0$ So the ratios $|\frac{z_1}{z_0}|^2 + \ldots + |\frac{z_4}{z_0}|^2 < 1$. Then $\mathscr{M}_5 \to \mathbb{C}H^4/\Gamma$, where Γ are five by five matrices in this ring where $\bar{M}HM = H$ and H is the diagonal map mentioned above. We can call Γ by $PU(1,4,\mathscr{E})$.

For surjectivity you have to extend f to some compactification. When you degenerate to a node, you get an A_2 singularity. If you get an A_2 singularity, you get infinite monodromy and a cusp in the Takaki compactification.

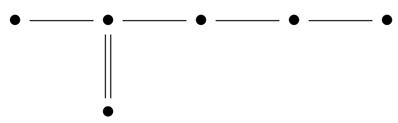
The monodromy when you approach a node is a complex reflection. In the normal direction it's a multiplication by $-\omega$. There's a hyperplane arrangement with some features. Whenever these intersect, it's at right angles. With A and B, it turns out that they either braid (intersect at infinity) or commute or don't interact. One thing this tells you is that this is a big group. What about injectivity? We all know what theorem we have to use. It's Clemens-Griffiths-Torelli, which is the big gun that makes this work. I should say one other thing about this arrangement.

$(\mathbb{C}H^4 - \mathscr{H})/\Gamma$ is what you think about for smooth spaces. Daniel Allcock proved that $\mathbb{C}H^4 - \mathscr{H}$ is a $K(\pi, 1)$ so you know some things about group homology.

I think that's all I wanted to say about, let me say a couple more things about what's known.

I knew about this when I was a graduate student except I didn't understand it. Kohn, do you have a watch that I could use? I didn't have time to look this up, but I think it goes back to like 1898 Actae. One thing that's fair to ascribe to Picard, probably, is, suppose you're interested in the moduli of four points. You can also say, maybe, $\{0, 1, a, \infty\}$. Picard had other fish to fry. You put $y^2 = x(x-1)(x-a)$ so you get a map to $\mathscr{H}/\Gamma$. Picard also put those in standard position. One answer to what the moduli space is, is, look at the complement to Δ and the 0 and 1 sections over a and b. You can also look at $x(x-1)(x-a)(x-b)$. I must have gotten something out of this paper as a graduate student. To a and b we get a marking $[mH^{1,0}_{\omega} \subset \mathbb{C}^3] \in \mathbb{C}H^2/Gamma$, where Γ is $PU(1,2,\mathscr{E})$.

One can also use this picture to study real cubic surfaces. I'll say what the picture is. In the old days, you can, there are five kinds of cubic surfaces. $\mathbb{RP}^2$ or $\mathbb{RP}^2$ with up to three handles. Then $\mathbb{RP}^2$ disjoint sum S^2, which is like $\mathbb{RP}^2$ minus a handle. This is $\mathbb{R}H^4/\Gamma_n^-$ where the group can be defined. $\Gamma_1^- = PO(-1, d_1, d_2, d_3, d_4)$ with Coxeter diagram



Now $\mathscr{M}^{\mathbb{R}}$ is the quotient by a group which is not not arithmetic. The volume is $37\pi^2/1080$. In principle we could do this with real cubic threefolds.

Let me finish up by talking about cubic threefolds. I want to concentrate on what's different. In one sense there shouldn't be too much to say. The main result is the Clemens-Griffiths-Torelli theorem. There's an addendum. The moduli space has dimension ten, but it goes into a fifteen dimensional space. What is the result? Bob, you and your students have worked on that, right?

[[unintelligible]]

Let X be a cubic three-fold and Y a four fold. You get a one dimensional space of $H^{3,1}$ and $H^{1,3}$ and then a twenty dimensional $H^{2,2}$. Then the ω eigenspace, I believe has one dimension of $3,1$ and ten of $2,2$. If you push this into $\mathbb{C}^1 1$ you get it inside $\mathbb{C}H^1 0$. The same type of arguments give you a map from smooth three-folds to $\mathbb{C}H^1 0/\Gamma$. To characterize the image you need to compactify the whole situatian. And so the best thing is to put in some simple singularities, but it turns out to be a little harder than that, you have semistable three-folds. Some of them are cusps, but then there's also the special case of the chordal cubic, the secant variety $R = RNC$. It has a nice equation $\begin{vmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \end{vmatrix} = 0$. It turns out f has a base point there. You blow up which shows that this indeterminacy is really there.

The idea is we have a family of threefolds degenerating to the chordal cubic CC Then you have the four folds degenerating. We need to calculate monodromy. You want the Hodge limit. No one can calculate these limits. You have to find the semistable model. Then again you need the Clemens Schmid sequence. I want to let $\hat{Y}_0 = Y_0' \cup Y_0''$ (a weighted blowup along a normal curve). It's a suitable weighted projectivization of $u^2 + v^2 + w^2 + z^3 + t^6 = 0$. If you map $[u, v, w, z, t] \to [w, z, t]$ which is a weighted $\mathbb{P}^2$, it's $\mathbb{P}^3[3, 2, 1]$. Usually you have a conic, you have a conic bundle over the weighted $\mathbb{P}^2$, which has cohomology $H^1(E)$ which has this symmetry as well.

Let me try to bring this to a conclusion. After doing some work, the limit Hodge structure consists of $\mathscr{E}$ and then another structur from $H^4(Y_0'')$. Let me write down the statement. The last factor is isomorphic as a certain kind of [unintelligible]structure to a Hodge structure coming from a Riemann surface. It's $H^1(C_B)_{(6)}$. The rational normal curve is R, and the pencil it came from is $F + tG + X_6^3 = 0$. The G determines the pencil, it's the normal direction. It cuts out a set of twelve points, called B. Let C be a six to one covering branched along the twelve points. It gets a Hodge relationship entirely from knowing where twelve points are. The limit Hodge structure is determined by twelve points on R, by zero dimensional data. As you vary the points around the map varies so it couldn't have been extended, you had to blow up first.

Twelve points on the Riemann sphere is nine moduli, this is correct. It's a ten dimensional moduli space and a nine dimensional divisor. To make anything work you have to use a Torelli theorem.

There are differences, the hypersurfaces can intersect, the nodal ones, at an angle $\phi = \arccos \frac{1}{\sqrt{3}}$.

3 MacPherson

It's a great pleasure. I wanted to begin with a historical review of intersection homology. This is joint work with Goresky long ago.

Consider a stratified space. It's divided up into submanifolds in a nice way, and the idea is, I won't go into technicalities, but if I take a given submanifold and look at a little neighborhood, it will have one which is a cone bundle on it. There's a point stratum, a line stratum and a surface stratum in the picture I've drawn. Assume that X and all its strata are even dimensional, like for example a complex variety.

There's a definition of intersection homology which is some i-cycles modulo some $i+1$-chains. The important thing is the fundamental calculation. Suppose I'm at a point stratum. A neighborhood will be conelike, it will have a conelike neighborhood C with boundary link δC. then $IH_{i+1}(C, \delta C) \to IH_i(\delta C)$ is an isomorphism for $i \geq n$ and $IH_i(C, \delta C) = 0$ for $i < n$. So somehow the intersection homology embeds in the intersection homology of the link, isomorphically for high degrees and zero for low degrees. This is really the key point, and now you can make up your own definition. If you have these two things be true, and analogous statements about cone bundle neighborhoods, then you will get intersection homology.

This intersection homology satisfies global Poincaré duality. When we found it we talked to Morgan and Sullivan right away. Since the conference is on John Morgan, I'll talk about his contribution. Suppose you have a manifold with boundary, and you look at $H_{i+1}(M, \delta M) \to H_i(\delta M)$ then the image A of this connecting map δ_* is equal to $A^\perp$ for the intersection pairing on M.

John's philosophy is that this is very much like the picture of the cone. The image of the cone is a self-annihilating subspace.

We chose the n and above dimensional self-annihilating subspace. This gave an intrisic explanation. One consequence of John's philosophy, in work of Banagl, well, we worked in even because of something about the middle stratum. You'd take half of the thing in the middle level. That turns out to be well-defined, and Banagl has shown it to be interesting and have topological applications.

I want to discuss a similar endeavor for locally symmetric spaces. Consider a complete negatively curved locally symmetric space X. If I have any point in a Riemannian manifold there is this reversing map on it, where a geodesic mapped through it, takes a point to its reflection through its geodesic through the point. This should be an isometry locally.

I should say I'm only interested in finite volume. Self-isometries are rare in general, so we consider multivalued self-isometries, a pair $s, t : C \to X$. Such a thing is called a Hecke correspondence, which is the same as saying X is arithmetic, so that if $\tilde{X}$ is a simply connected covering space, a Lie group because X is locally symmetric, $\pi_1(X)$ is an arithmetic subgroup of X.

The project, which in a way I'll mention later, is part of the Langlands program to find a Lefschetz fixed point theorem for C. This is a correspondence, it should induce a map on something. This is doomed from the beginning because X is noncompact. Consider $x \mapsto x+1$ in $\mathbb{R}$. The Lefschetz number of f is $\sum(-1)^i \mathrm{tr}\ f_* H_i(X) \circlearrowright$ so this would be one, but it's supposed to be the sum of the number of fixed points.

When you compactify this, you get $L(p) = 1$ for positive infinity and 0 for negative infinity.

These locally symmetric spaces have tons of compactifications. There's one that they all have, called the reductive Borel-Serre compactification $\bar{X}^{RBS}$.

[Missed a bit]

δC is a nilmanifold, you can do cohomological calculations on them. You want a cohomology theory adapted to this situation.

So we define, and by the way, I should have said, what I've been describing is part of an enormous project. It's joint with Goresky but some is joint with Harder and Kottwitz.

If you have a point singularity, everyone's favorite example is the Poincaré plane, if I take a space like this, and as everyone knows what i get is, near infinity it looks like a cusp, and the reductive Borel-Serre ads a point at the cusp.

There's a beautiful idea due to Looijenga, in this case, which I call the Looijenga Hecke operator or correspondence, which is, take the space and move it off to the right some distance. You see, it doesn't fit itself anymore. If I do it the right amount I can cut along a geodesic and wrap it around itself several times.

This, by the way, is in some sense a characteristic zero analogue of Frobenius. There are few examples in mathematics of a geometric or topological picture of Frobenius. So then I can use geodesic flow which is not an isometry, to flow back. Now, the idea is to look at the homology of the link and make a picture like John Morgan's picture but use instead the weight of this map. So look at the weights acting on $H_*(X)$. So I can draw a weight diagram like

and then make a group A which is just the top half. So $WH_*(C, \delta C) \cong A \subset WH_*(\delta C)$. The weight and degree are very different so you're doing quite a different truncation.

In the case where there's nothing in the middle this will satisfy Poincaré duality.

[Dennis: This is a local thing, so what will satisfy duality?]

Global weighted homology. The proof is rather hard but the intuition is the same, it's as if you're adding a manifold with boundary to another. That's the intuition behind the Poincaré duality.

Okay, at this point to advance the theory more I need to do something very sad, assume X is Hermitian. It might be a principally polarized Abelian variety, as in Phillip's talk. Let me say why this is sad.

Langlands' conjectures say, the Hecke correspondences should have profound arithmetic consequences, something about motives and Galois representations. No one has any idea how to do such a thing. The program is to try to do this when you have a Hermitian symmetric variety. I want to make this point with this audience in particular, because this is the first place where this happens, in a three manifold.

[Could you give an example of an arithmetic question about hyperbolic three manifolds?]

There are incredible computer experiments, where you look for correspondences between particular three manifolds and particular motives, L-functions, like, traces of Frobeniuses acting on an algebraic variety or the trace of Hecke correspondences acting on weighted homology of a three manifold.

Okay, so anyhow, the Hermitian symmetric case is still very interesting and not quite finished. So what we proved jointly with Harder, you have two compactifications, the reductive Borel-Serre and then the [unintelligible], which only exists in the Hermitian case. It's no easier to compute things about the link than about X itself. Our result is that $WH_*(\bar{X}^{RBS}, L) \cong IH_*(\bar{X}^{BB}, L) \cong H^{2n-i}_{(2)}(X, L)$, for L an appropriate local system. [I left out some history.] Arthur as part of his thousands of pages of work on the trace formula, was abele to give a formula for the trace acting on $H^{2n-i}_{(2)}$ and you can transport those back, and we have an interpretation or reproof of his formula in a geometric manner. This is one of the steps that is needed in the Langlands program.

What's different about those two fixed points? The map was contracting on one and expanding on the other.

[Dennis: you talked about local things and then you switch back and forth to global things, I don't understand which is global. Are you assuming C is global?]

We have $L(C) = \sum (-1)^i \mathrm{tr}\ \bar{C}_i WH_i(X) \circlearrowleft$

The things that contain the beautiful arithmetic juice are the global things. We understand the local very well.

I should have said, there are several Looijenga correspondences, I'll get the same group A, all that will happen is an expansion and contraction of the picture.

Suppose we had a map f that was hyperbolic. So in some sense you have expanding and contracting directions. To make this precise you have to explain what this means at a singularity. At one point I have a fixed point. This is the sum over fixed point components

K of $\mathscr{L}(K)$. Grothiendiek proved that there is such a thing but he didn't define it. You take a box around K modulo its outgoing edges, it's $\sum(-1)^i \mathrm{tr}\ f_{*i} W H_i(B, C)$. Because of hyperbolicity and excision this is well-defined.

Let me just finish by mentioning, there has been a remarkable development, Sophie Morel, in a thesis she's completing, has extended this to an algebraic theory. The reductive Borel-Serre compactification is not an algebraic variety, while the other one $[BB]$ is. So the first thing she had to overcome was, she had to overcome the fact that this isn't an algebraic variety. Also, our definition of hyperbolic was topological. Her formula is true for composition of Hecke correspondences and higher Frobenius. Okay, I'll stop.

[How does Jim Arthur think about this?]

There are magic directions in a locally symmetric space. There's first of all a flat subspace and then some nice directions. He has things growing and increasing in these directions. He'll chop up a space into pieces. Then you'll add a point for this piece and this piece and a line for this piece. The first thing he does is chop things up into certain subgroups.