Morgan Conference
May Day, 2006

Gabriel C. Drummond-Cole

May 1, 2006

I would like to welcome you all. I have a couple of announcements. The talks by Gabai and Minsky are reversed. On Wednesday, the slot, Ron Stern will now be talking 5-6 and Khovanov will be speaking at 11:30. Those are the changes that we know. The last thing is, if anyone has not yet signed up for the banquet, we need to know today by two. Send to banquet@math.columbia.edu with just the number in the subject line. It's again a pleasure to welcome you all. It's a pleasure to introduce Richard Hamilton.

1 Hamilton

Well, happy birthday, John. It's a nice time of year to have a conference. You should have had the conference start on Saturday, there was a Hawaiian luau complete with the roasted pig and girls dancing the hula outside the math building. As a result, I didn't prepare my talk very well.

It turns out that, I spent a long time with Harnak equations and [unintelligible]. This is something that comes up in heat problems. In Perelman's paper he has the heat equation and then a different Harnak on the heat equation. On flat space you can forget the Ricci flow and you get a hyperbolic inequality you can integrate along paths, and that gives you the transport equation.

The estimates at first order are equalities and on the Laplacian are inequalities.

This will be joint work with Cao H-D and Deskalopoulos, and Sesum. The basic situation is that we have a solution to the Ricci flow where $\frac{\partial}{\partial t}g = -2R_c$. and $0 \leq t \leq T < \infty$. Think of $\tau = T - t$ and think of the Ricci flow running backwards. You can't solve it backwards, but you can solve it forwards and reverse the time.

If you don't care about the Ricci flow, you can work on flat space. Any curvature on flat space is zero. So if you're only interested in PDE and not geometry this still might be interesting.

I want to look at the heat equation. Which will be $\frac{\partial u}{\partial \tau} = \Delta u$. I really want the adjoint, so I subtract Ru. It turns out that it's useful to look at log u. When I did the Harnak there was a matrix and I didn't want to take a logarithm. As a scalar you write $u = (4\pi\tau)^{-n/2}e^{-f}$. If I think of this as $\Box u = \frac{\partial u}{\partial \tau} - \Delta u - Ru = 0$. Then I write $Pf - \Delta f + |Df|^2 - P + n/2\tau$ and $Ef = 2\Delta f - |Df|^2 + R + 1/\tau f - n/\tau$.

So $\Box(\tau Efu) + 2\tau|Rc + D^2 f - (1/2\tau)g|^2 = 0$. So if $Ef \le 0$ at $\tau_1 < \tau_2$ then $Ef \le 0$ at τ_2.

You can see thet where τ goes to zero, you don't really know it will be negative. It's hard to know when $Ef \le 0$. Where u starts as a delta function then Ef will be less than or equal to zero in the sense of distributions.

Someone tried to show it with heat kernels but couldn't do it, he used a different argument that was quite nice. When $Pf = 0$ and $Ef \le 0$ you can look at $Hf = 2\frac{\partial f}{\partial \tau} + |Df|^2 - R + 1/\tau f|$. So $Hf = 2Pf + Ef$. So what you get is if $Pf = 0$ and $Ef \le 0$ at time τ_1 Then $Ef \le 0$ for $\tau > \tau_1$, which implies $Hf \le 0$. You can integrate this along paths.

The usual trick that you do is, write $\frac{d}{d\tau} f = \frac{partial}{\partial \tau} f + Df \frac{partial P}{\partial \tau} \le -1/2Df^2 + \cdots$

I'll just give the result

$$f(P_2, \tau_2) \le \sqrt{\tau_1/\tau_2} f(P_1, \tau_1) + \frac{1}{2\sqrt{\tau_2}} \int_{\tau_1}^{\tau_2} \sqrt{\tau}((\frac{dP}{d\tau})^2 + R) d\tau.$$

An ℓ-function satisfies $Hf = 0$.

You can propagate that to future time by propagating along characteristics. If I take $V = D\ell$ then the characteristics give a system for (P, ℓ, V) where $\frac{dP}{d\tau} = V^*$, $fracd\ell d\tau = 1/2|V|^2 + \frac{1}{2}R + \frac{1}{2\tau}\ell$, and $\frac{dV}{d\tau} = DR - \frac{1}{\tau}V$. You can then realize this solution by a minimizing path integral, and I'm sadly lacking in my physics education, but I'm told that this is Hamilton Jacobi theory. That's not me, that's the real Hamilton. You get, given $\ell(P, \tau_1)$ at time τ_1 the solution $H\ell = 0$ is given by $\ell(P_2, \tau_2) = \inf\{\sqrt{frac\tau_1\tau_2}\ell(P_1, P_2) + \frac{1}{2\sqrt{\tau_2}} \int_{\tau_1} \tau_2 \sqrt{\tau}((|\frac{dP}{d\tau}^2 + R)) d\tau\}$. where the infimum is taken over paths P_1.

If you've read Perelman's paper, what he has is a little different. He fixes P_1 and has the function coming out. That's equivalent to taking ℓ as zero at one point and infinity everywhere else. You only need to know what you are anywhere else. Thinking of it as a solution to a transport equation this is a very obvious generalization.

Let me say a word here that as we know with hyperbolic equations, you don't expect the solution to be smooth. In fact, when I say I'm propagation ℓ, I'm taking a weak solution with a minimality property.

There's another way to think of this, if you look at all the other paths, what you find is, you look above $M \times [0, T]$ at the one-jet bundle $J^1 M$. What's going on, down below you have the point p, τ while above it's (P, ℓ, V, τ). What's going on, when you specify ℓ at τ_1, that gives you a submanifold, the graph of the one-jet of ℓ at that time.

There's a natural contact structure and the submanifold has it vanish there. I think this is a Legendrian submanifold. Anyway, what you get is, you have the submanifold and you can propagate it as a submanifold. Then each point follows by the characteristic. You get multiple sheetings if you project down. Just in order to see how this works, I had Muffin do an example. This is a virtual mathematician living in my Macintosh G4.

I took a function which initially had a W-shaped graph, and then propagated this forward. If you look at the equations, you don't get any real bad growth. At worst it's exponential in V. So grad ℓ stays bounded but ℓ is not a smooth graph. At $\tau = 4$ here I get a picture with two cusps and a crossing. It's a singular projection of something embedded in the jet bundle. I think it will stay smooth above. What's the difference between this and the infimum? You wouldn't see the overlap here.

What's interesting here is, Perelman starts with his ℓ-function concentrated at a single point and goes through an elaborate argument to get it to satisfy an elliptic inequality. He defines it as above, as the infimum but with fixed P_1, Then $E\ell \leq 0$. This is the same as in Harnak.

The argument here is completely solid but difficult. I started thinking, how does $E\ell$ evolve? Along a characteristic, $\frac{d}{d\tau}E\ell + 3/(2\tau)E\ell + 2|D^2\ell + Pc - \frac{1}{2\tau}g|^2]0$. So if $H\ell$ vanishes and $E\ell \leq 9$ then $E\ell$ stays below zero.

The hyperbolic inequality gives the elliptic one, which then yields the parabolic inequality. You get $P\ell \geq 0$. If you write w as the logarithm, then $\Box w \leq 0$ and so $\frac{dw}{d\tau} \leq \Delta w - Rm$. So the integral of w is monotone decreasing.

The integral of w is what Perelman calls the reduced volume. I realized from Perelman making a similar statement at a point, that $wd\mu$ is monotone decreasing along characteristics as τ increasing, so the volume form has its own monotonicity.

What happens if you look at this reduced volume? Well, the bigger ℓ is, the less the volume you get. So upstairs where ℓ is bigger you get less volume. I think that inculding the extra info gives iou something stronger.

Perelman has shawn us a bunch of toys. I don't know what to do to them, I don't read the instructions, I just play with them, play around.

For those of you who haven't seen it, you get this wonderful non-collapsing result, which is, given a solution to the Ricci flow on $0 \leq t \leq T$, there exist $r_0, \kappa \geq 0$ such that if $R \leq 1/r^2$ in $B_r \times [unintelligible]$ then $Vol \geq \kappa n^m$.

This rules out cigars. Then when you go out aways and take a ball, you get not much curvature or much volume, which contradicts noncollapse.

With the ℓ-function this follows from monotonicity of the reduced volume. You start at a small τ and propagate to T.

I'm short on time but I want to say one other application. That is in the K ahler case. Look at the canonical case, positive Chern class. I'll assume that $[g]$ is $[Rc]$ maybe with a constant. Then the solution exists for a maximal time interval and as you approach maximal time the volume goes to zero. There's recent work of Perelman, using the arguments of the sort that you have on the ℓ function, anrd what you find is, the curvature is bounded by constant over time to blowup. Yuo get bounds on the scalar curvature and the diameter using time to blowup.

To what extent does this extend to the blowup? Since I've had my pictures with τ as time, then my actual time is flowing back. So now the natural thing to do is to write down the ℓ-function as an infimum. When everything collapses, I'll call this the apocalypse. If I introduce Biblical terminology, maybe I'll get funded. If you lived in this world I'm sure you'd see the moon turn into blood and the stars fall.

I take any path and this becomes the apocalyptic ℓ-function. This is a naturally defined function and it satisfies the transport equation.

As a theorem, $E\ell \leq 0$. Then you can take a sequence of times going back to zero and start with it being a suitable negative constant so that $E\ell \leq 0$. Then because the R is bounded by a constant over τ.

You can then go back to the f and come to a time and say $f_k = \ell$ and so $Ef\kappa \leq 0$. So u satisfies the affine heat equation. u is uniformly bounded below. Among solutions to the parabolic equation, this one is the smallest. This is a unquely defned solution.

There's also a question, look at the manifold in the jet bundle, if it's a manifold.

I thought I'd show you some of these ideas. There's a lot to play with. Then we'll stop there. The next talk will start at 11:05. That will be Dave Gabai.

2 Gabai

I do not take notes for lectures with transparencies.

3 Minsky

John's work was very influential to me as a graduate student. Some of the concepts, tiny shadows of them, will show up in this talk. I want to talk about the mapping class group of a surface. Let S be a surface. It can have boundary or not. Then $MCG(S) = Homeo_+(S)/Homeo_0(S)$. It's a finitely generated group. This is joint work with Masur and Behrstock.

So from a mapping class group you can get a Cayley graph. I want to look at the geometric subgroups. I don't know if that's a good name. Think of the subgroups, say, that stabilize a subsurface. Suppose that Δ is a finite union of curves in S. Then $Stab(\Delta)$ is a subgroup of the mapping class group. I want to understand how these sit in the group, how they intersect one another, and so on.

$\Delta = \{\delta\}$ could be a simple closed curve. Then the stabilizer of Δ lives in a short exact sequence, so

$$0 \to \mathbb{Z} \to Stab(\Delta) \to MCG(W - \delta) \to 0$$

These are called Dehn twists. All you can do is cut and reglue. You can now understand this in terms of an extension of a lower mapping class group.

But you can say, if you're interested only in coarse geometry, that $Stab(\delta)$ is quasiisometric to $\mathbb{Z} \times MCG(S - \delta)$. This means "up to multiplicative and added distortion." Furthermore, $Stab(\delta) \hookrightarrow MCG(S)$ quasiisometrically. So it's undistorted, not curved too much, in the big group. You might think, you take n steps in the subgroup, there might be a shortcut in the big group, but it's a quasigeodesic. This is a true statement.

Special cases are, if $\Delta = \{\delta_1, \dots, \delta_p\}$ are disjoint simple closed curves. Then similarly, $\mathbb{Z}^{\Delta} \hookrightarrow MCG(S)$ quasiisometrically. This was Mosher for $\delta S \neq 0$ and Farb-Lubotzky-Minsky in general.

I want to give a distance formula for $MCG(S)$.

Pick μ, a binding set of curves. You can make a picture where you've cut the surface up into disks. The stabilizer of μ is finite. You can think about the orbit, in the set of all pictures, of μ. So I can look at the pictures I get by applying mapping class group elements to the picture.

Theorem 1 *Masur, Minsky*

$$dist(\beta, \gamma) \cong \sum_{W \subset S \text{ essential}} [d_W(\beta\mu, \gamma\mu)]_K.$$

Here $[\ ]_K$ is the threshhold function which takes $x < K$ to zero and $x > K$ to x. If μ and ν are curve systms and $W \subset S$ is a subsurface. I want to measure their relative complexity inside the subsurface. So $d_W(\mu, \nu)$ is the minimum n such that there exists a sequence $a_0, \dots, a_n$ of essential arcs. with a_0 in $\mu \cap W$, $a_n \in \nu \cap W$, and $a_i \cap a_{i+1} = \emptyset$. This is essentially distance in the curve complex of W. That's some number associated with a pair of pictures and a subsurface. It's also slightly different when W is an annulus.

So the theorem says the numbers, up to multiplication and addition of constants, are the same. The correct statement is, there exists a $K_0(S)$. Then for all $K > K_0$ there exist numbers for which the estimate holds true.

So this is actually very hard to compute, both of these.

Theorem 2 *(Masur, Minsky) $C(W)$ (the curve complex) is δ-hyperbolic. It gives you something like an embedding of $MCG(S)$ into $\prod_W C(W)$. Behrstock studied this in his thesis. It's a huge space, a product of infinitely many hyperbolic spaces. It turns out that the image lives inside a very thin set in this product.*

The point is that distance in $\mathbb{Z} \times MCG(W - \delta)$ can be approximated by taking a sum over the surfces which do not cross δ. Another way to say it is, it gives you a kind of product decomposition.

A slightly fancier version of the same idea says how two of these intersect.

I can ask what the union of $Stab(\delta)$ with $Stab(\delta')$? What about the intersection? Call W the surface filled by δ and δ'.

I want to talk about some related ideas. One thing that we end oup talking about is Abelian subgroups. You can ask what are the Abelian subgroups, this was Birman-Lubotzky-McCarthy. One way to da this is to take Δ a maximal set of disjoint simple closed curves, then $\#\Delta = 3g - 3 + b$. Then $Stab(\Delta) = \mathbb{Z}^{\Delta}$. You can make, if you choose Pseudo Asonov elements; I should name check someone and mention Thurston's classification into finite, pseudo-Asonov, and ***

Cgis next question is due to Bruck-Fadr This process was giving $\mathbb{Z}^n$. Suppose wie ask, suppose $\mathbb{Z}^n \hookrightarrow MCG(S)$. This is quasiisometric but it is not a homeomorphism. Can you do this?

Theorem 3 *U Hammenstadt, Behrstock-M.*
Yes.

Let's rescale the picture and see what we can say.

Suppose you have a quasiisometric embedding of $\mathbb{Z}^n$ into the mapping class group. You can multiply the picture by a small constant ϵ. Both of these are quasiisometries. If you could let ϵ go to zero and take a limit, you wolud get what? You see what it converges to. In our case it's $\mathbb{R}^n$. The mapping class group converges to something, $\mathscr{M}$, the asymptotic cone, and the map in the limit is biLipschtz.

So what? Now you can use topology.

Theorem 4 $\widehat{\dim}\mathscr{M} = 3g - 3 + b$.
widehat$\dim(X) = \sup\{$ *covering dimension of A where A is a locally compact subset.*$\}$

The MCG completes with this "ultralimit" to something, the space of all possible limit pictures.

If S is $T^2, T^2 - pt$. Then $MCG(S) = SL_2(\mathbb{Z})$. Then in the limit you have a tree with infinite branches. This is called an $\mathbb{R}$-tree. You take the asymptotc limit and get out of algebra. Any point separates, it's dimension one. You can't embed $\mathbb{Z}^2$ quasiisometrically in $SL_2\mathbb{Z}$ because then in the limit you could embed $\mathbb{R}^2$ quasiisometrically in this $\mathbb{R}$-tree.

What about $S_{1,2}$. The number of Δ is two. So pick a curve. There are two choices, into a three-holed sphere and a one holed torus, and there's another choice which makes a four-holed sphere. So $Stab(\delta) \cong \mathbb{Z} \times SL(2, \mathbb{Z})$. Then as ϵ goes to zero I should get $Q(\delta) \overset{\text{homeo}}{\cong} \mathbb{R} \times T$. I want to show that there are no $\mathbb{R}^3$ in this space. I'm going to need to say something global. This comes out not just of the distance formula, but that circle of ideas.

Lemma 1 *There exists a $\pi_\delta : MCG \to \mathbb{Z}$ and b, c such that on $Stab(\delta)$, $\pi \cong$ projection to the first factor, and such that if $\beta \in MCG \backslash Stab(S)$ and $R = dist(\beta, Stab(\delta))$ then $diam(\pi_\delta(B_{cR}(\beta))) < b$.*

What good is this statement, it has quantifiers and constants, and so on.

In the ultralimit we got $\pi_\delta : \mathscr{M} \to \mathbb{R}$ such that $\pi_\delta|_{Q(\delta)}$ and π_δ outside $Q(\delta)$ is locally constant. This is nicer to deal with because as a topologist I know how to work with constant functions. As a consequence, if you look at a vertical fiber, $\{s\} \times T$ this separates $\mathscr{M}$. If I chose points on opposite sides to connect, removing such a fiber cuts into already three components. Suppose that $E \subset \mathscr{M}$ and $E \cong \mathbb{R}^3$. Then suppose $\pi_\delta|_E$ is nonconstant. Then you can find an S between two points with different values under π_δ. Then I get a one dimensional set $E \cap \{s\} \times T$ which separates E, a contradiction.

I've used up all my time. I claim there are enough projections like π_δ to separate points in $\mathscr{M}$. If you allow any δ or a limit of those, you get a lot more projections. This discussion corresponds to the projection on the first factor. You can try to project to $SL(2, \mathbb{Z})$, which you cannot. But you do something similar and eventually you show you cannot have an embedded $\mathbb{R}^3$. In general you use induction, it's harder.

4 Tian

It's a great honor to speak here. I remember John proved in a course or talk that higher Massey products vanish on K ahler manifolds. It gave me some sort of direction for things I did with Calabi-Yau manifolds.

The problem I will talk about today has interested me for a long time. What I am going to talk today, I was interested in Lefschetz fibrations and I was trying to use harmonic maps. Pseudo-holomorphic curves seem to be more powerful right now.

For today's talk X will be a symplectic 4-manifold with $c_1(X) > 0$, such as $X = \mathbb{CP}^2, S^2 \times S^2$, or $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. Del Pezzo surfaces. Let ω be a symplectic two-form $\omega^2 \neq 0$ and $d\omega = 0$. Then $\Sigma \subset X$ is symplectic if for all $p \in \Sigma$, $\omega|_{T_p\Sigma} > 0$.

The isotopy problem is, given two symplectic Σ_i which are are homotopic to each other, is there a family of Σ_t joining them. If you don't impose this condition, the symplectic condition, it's easy to find a counterexample.

In general, say Σ may have nodes and cusps. I just mean, it's not too far from complex singularities. Let me say, first, why don't you ask the same question for other X? If you allow it to be too general, the answer is no.

1. There are examples from Fintushel and Stern, there exist X a four-manifold with homotopic but not isotopic Σ_i.
2. Moishezon-Auraux, it is also no if there are too many cusps relative to the degree of the surface.
3. $X = \mathbb{CP}^2$, Gromov, and d the degree. If $H_2(X, \mathbb{Z}) = \mathbb{Z}$ and $d \in \{1, 2\}$, yes. Sikarov did it for $d = 3$ and Shevchishin up to six. Siebert Tian brought it up to seventeen. This is yes for the embedded case. If you allow for cusps, it was first studied by Ono. I never saw a preprint but he told me in a private discusion. Also S. Francisco, a cubic surface with one cusp is always isotopic to a holomorphic one.

The motivation is, $\pi : Y \to S^2$ is a Lefschetz fibration if for all $b \in S^2$ there exists a neighborhood U such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d] & \text{Holomorphic Fibration} \arrow[d] \\ U \arrow[r, "\varphi"] & D \subset \mathbb{C}^1 \end{tikzcd}$$

and all singular fibers. It has only nodes as singularities.

Gompf, Donaldson have that symplectic four-manifolds up to blowup correspond to Lefschetz fibrations. Aubaux, branched coverings.

Locally they are holomorphic but globally they are patched together.

In most cases this is given by, the fibration $\pi : Y \to S^2$ has the structure, this leads to a branched covering of X a rational ruled surface, and π a $d : 1$ covering. You can make this pairing sufficiently well so that if $B \subset X$ is the branch locus and what is known is that B is symplectic with respect to some ω and secondly B has finitely many, has only nodes and cusps. Further, $\{Y \to S^2\}/\sim$ is determined by $\{B\}/\sim$.

If $d = 2$, this time $p : Y \to X$ is a double cover. Generic fibers, actually we prove the following theorem.

Theorem 5 *(Siebert, T.)*
Let π be a genus two fibration with irreducible singular fibers. Also assume that the monodromy group generates the mapping class group of a generic fiber. That is equivalent to saying the branch locus is irreducible. Then Y is equivalent to a holomorphic fibration.

The first condition is necessary, or there is a counterexample by Ozbeacci and Stipsicz. If the genus is two then $Y \#_{\text{fiber}} Y_1$ (algebraic rational surface) then this is a holomorphic surface. This is Auraux.

The problem with irreducibility is that you won't be able to do something, and if you make it smooth it won't have a positive first Chern class.

Conjecture 1 *Seibert-T.*
Any hyperelliptic Lefschetz fibration without an irreducible fiber is holomorphic. In general one may expect, if $d \geq 3$, *then any Lefschetz fibration is of the form* $Y_1 - Y_2$, *with those two holomorphic surfaces. That is,* $Y \# Y_1 = Y_2$.

This is equivalence to uniqueness of isotopy classes of, look at branch locuses in X which is rational ruled [unintelligible]

Theorem 6 *IF* π *is a rational ruled surface and* Σ *is smooth then if the degree of* π *with respect to this surface is less than or equal te seven thon* Σ *is equivalent to a holomorphic.*

Let me go on to first try to generalize this conjecture. The cusp condition, the conjecture is, if Σ has m cusps, and $c_1(X)\Sigma > m$ then Σ is isotopic to a holomorphic curve with m cusps. I had my student working on this, she got it to $3m$ and then went to work in finance.

Then you can ask the simple question, is there a Lefschetz fibration such that B has cusps less than $c_1(X)B$? It turns out that this is not easy. I went to check some example, it's with Y in the third product of $\mathbb{CP}^1$ defined by a polynomial in $s,\ t,\ 1$. This is a three to one covering. If you change by any number bigger than one you don't have this.

It looks like a hard problem. In general you can make it formalized. You can do the following things. Let's say, given $d \geq 3$ and $g \geq 2$. You can define $B_{d,g}(X)$ as the supremum of $\frac{c_1(X)B_Y}{\#cusps}$ such that it's a generic $d:1$ covering $p: Y \to X$ such that genus is g.

It's known $B_{d,g}$ is bigger than one but it seems that this is the best upper bound. If you make $g \to \infty$, the difference goes to zero

I want to find an algebraic surface. If I find the right kind of thing, I can make a connect sum and make the covering behave better.

Now let me say what I know. I did not have time to prove this in the most general case. But this I can prove.

Theorem 7 *Let* $X = \mathbb{CP}^2$, *no, a rational surface with* $c_1(X) > 0$. *Any symplectic surface inside* X *with* m *cusps* ($m < c_1(X)\Sigma$ *and of geometric genus zero is isotopic to a holomorphic curve.*

The proof is the following. It was studied by Gromov. first find an almost complex structure J on X such that ω is tamed, such that Σ is J-holomolphic. Then by Gromov, there is path J_t with $J_0 = J$ and J_1 integrable. Let $M = \{u, J_t$ such that $u : S^2 \to X$ is fixed topologically, u is J_t holomorphic, $u(S^2)$ has exactly m cusps, and $u(S^2)$ contains fixed points x_i.

As a lemma, M is smooth. The canonical projection is regular.

Next by transversality, and compactness, J_t is generic then M is compact.

Let me say where the difficulty arises. I want to state a local isotopy teorem. In the immersed case by Shevchizhin, given J let C be a J-holomorphic curv in $B_1(O) \subset \mathbb{R}^4$. with isolated singularity at 0. Suppose that $c_1,\ c_2$ are closed to c. So $c_1 c_2$ has the same number of cusps. Then c_1 is isotopic to c_2.