Link Homology
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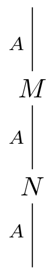
I need to start by talking about bimodules and Hochschild homology. So fix a ring A and we have M an A-bimodule. Depict this as M with lines from the top and bottom signifying the left and right action. You can tensor bimodules by sticking them together pictorially.



So eventually this will be a derived tensor product. If M is a right A-module and N is a left A-module then the usually tensor product $M \otimes N$ is an Abelian group, but the derived tensor product exists, $M \hat{\otimes}_A N$ is: take a projective resolution P_i of M, tensor every term with N, and take homology. This is a graded Abelian group. We can do the same for bimodules. If we do it for bimodules, we can resolve with projective bimodules. Thriftily we can choose the resolution to be of bimodules projective as right A-modules. If M is right projective then the derived tensor product is just the usual one. This is completely unsymmetric in M and N.

Now we can close things off by looking at $M/am - ma$, symmetrizing by setting the left and right actions equal to one another. These are the A-coinvariants of M. You run into the same trouble because this functor is not exact, it's right exact. Typically quotient functors are right exact, subobject functors are left exact. We can take a resolution of A by projectives. That's because $M_A = M \otimes_{A \otimes A^{op}} A$ So we convert A into a complex of projective $A \otimes A^{op}$-modules, and you get what we call the Hochschild homology of M; $HH(M) = H(M \otimes P^*)$.

So say $A = \mathbb{Q}[x]$. Then this is $\nvdash \to Q[x] \otimes \mathbb{Q}[x] \to \mathbb{Q}[x] \otimes \mathbb{Q}[x] \to \mathbb{Q}[x] \to 0$. The differential here first takes $1 \otimes 1$ to $1 \otimes x - x \otimes 1$, and the next one is multiplication.

When you tensor you get $0 \to M \to M \to 0$ with the map $m \mapsto mx - xm$. So $HH_0(M) = M_A$ and $HH_1(M) = M^A$, the coinvariants and invariants. Usually this would not be so nice, M^A is defined as $HH^0(M)$ but this will not always be $HH_1(M)$.

What about for $A = \mathbb{Q}[x_1, \ldots, x_n]$? We can tensor together the last resolutions. We can take $A \otimes A \to A \otimes A \to 0$ where $1 \otimes 1$ goes to $x_i \otimes 1 - 1 \otimes x_i$, and tensor n copies of this together. So we get 2^n copies of $A \otimes A$ in a complex. So for HH we need to tensor with M giving 2^n copies of M taking $m \to x_i m - m x_i$. In this case it will be $HH_0(M) = M_A$, the coinvariants, and all the way up to $HH_n(M) = M^A$ the invariants.

Now let $A_i \subset A$ be the subring of polynomials invariant under the permutiation of x_i and x_{i+1}. As an A_i-module, this is free of rank 2, with $A = A_i \cdot 1 \oplus A_i x_i$.

So take $B_i = A \otimes_{A_i} A$, this is projective as a left and as a right A-module. We'll keep track of grading, giving each x_i degree two. I'll explain why two and not one later. Then A_i is a homogeneous subring, and B_i is a graded bimodule. There is a bimodule map $B_i \to A$ taking $a \otimes b$ to ab. This is a complex of bimodules, $0 \to B_i \to A \to 0$. There's another like $0 \to A \to B \to 0$ which takes 1 to $(x_i - x_{i+1}) \otimes 1 + 1 \otimes 1 \otimes (x_i - x_{i+1})$. Just shift the degree of B_i down by two to make the complex differential have degree zero.

Take the simplest braids σ_i in the braid group Br_n; this is generated by σ_i which interchanges i and $i+1$. Assign the two complexes I've just generated to $\sigma_i^{\pm 1}$. So what is $F(\sigma_i) \otimes F(\sigma_i^{-1})$? This is $B_i \otimes A \to B_i \otimes B_i\{-2\} \oplus A \to A \otimes B\{-2\}$. I wanted to choose the grading of my modules so that A is always in degree zero.

Notice that everything is over A so we get a simplification:

$$0 \to B_i \to B_i\{-2\} \oplus B_i \oplus A \to B_i\{-2\} \to 0$$

There is a differential which we didn't write down. If you want to write down what ∂ is, you find out it decomposes into $0 \to B_i \to^1 \to B_i \to 0$,
$0 \to B_i\{-2\} \to^1 B_i\{-2\} \to 0$ and $0 \to A \to 0$. In the homotopy category, complexes of graded A-bimodules $F(\sigma_i) \otimes F(\sigma_i^{-1}) \cong A$. So tensoring two complexes of the generating braid and its inverse together yields the complex of the trivial braid.

Theorem 1 *$F(\sigma_i)$ gives rise to a braid group action on this (homotopy) category $\mathscr{C}$. This means $g \to F(g)$ and $F(gh) \cong F(g)F(h)$ and $F(1) = Id$. Eventually we want natural equivalences.*

We want $F(\sigma_i) \otimes F(\sigma_j) \cong F(\sigma_j) \otimes F(\sigma_i)$ trivially, and $F(\sigma_i) \otimes F(\sigma_{i+1}) \otimes F(\sigma_i) \cong F(\sigma_{i+1}) \otimes F(\sigma_i) \otimes F(\sigma_{i+1})$.

We've gotten to braids, but we want to get to links. To do this we take the closure of a braid. So $\sigma \mapsto \hat{\sigma}$. So closure, remember the beginning of the lecture, should correspond to taking the

Hochschild homology. So starting with σ we take $HH_F(\sigma)$. This has $\to F^j(\sigma) \to F^{j+1}(\sigma) \to$ where F_j is a direct sum of tensor products af B_i. This is additionally graded and the differential preserves the grading. If you start with a graded ring, the Hochschild homology is bigraded instead of being merely graded.

I will get a bigraded vector space $HH(F^j(\sigma)) \to HH(F^{j+1}(\sigma)) \to$, with each term bigraded. The differential preserves the bigrading. We get a complex of bigraded vector spaces. So now we can take homology again since $HH(\partial)^2 = 0$ since HH is functorial. So $H(\sigma) = H(HH(F(\sigma)), HH(\sigma))$.

Why do this?

Theorem 2 *$H(\sigma)$ is triply graded, depends only on $\hat{\sigma}$ and has Euler characteristic equal to the HOMFLY polynomial of links.*

The HOMFlY is uniquely determined by the conditions $\lambda P(L_+) - \lambda P(L_-) - (q - q^{-1})P(L_0) = 0$ and the value of $P(\text{unknot})$. For instance, the polynomial of the two component unlink is $\frac{\lambda - \lambda^{-1}}{q - q^{-1}}$. This should be expanded as a power series.

For example, σ is the trivial braid. Then $F(\sigma)$ is A and $H(L) = HH(A) = A \otimes \wedge(y_1, \ldots, y_n)$.

We can specialize. Take $\lambda = q^n$. This is a single variable representation. This can be described with $U_q(\mathfrak{sl}(n))$. These are one variable polynomials in $\mathbb{Z}[q, q^{-1}]$. The interesting cases are $P_0(L)$ the Alexander polynomial, $P_1(L)$ trivial, and $P_2(L)$ the Jones polynomial. Now there is a family of homology theories, bigraded, which lift these.

We have Oszvath-Szabo-Rasmussen, $P_0(L) = \sum (-1)^i q^j rk\ H_0^{i,j}(L)$ and a similar thing in my old work for the Jones polynomial. These look nicer than this work currently, as they are functors from the category of link cobordisms to some algebraic category. The objects are oriented links in S^3 and the morphisms are isotopy classes of surfaces in $\mathbb{R}^3 \times I$ with boundary the difference of the links.

The theories I have listed are functorial, at least conjecturally, in the Oszvath-Rasmussen-Szabo case, and up to a sign in the Jones case.

To define homology we needed a choice of braid, while in the other cases we can work with any planar projection of a link. We cannot hope that it is functorial under cobordisms, but there should be a reduced version without A which is still functorial.

We would ned the H of the unknot to be a Frobenius algebra. It's a commutative associative algebra and coalgebra with a unit and a counit. Any Frobenius algebra is finite dimensional. But $H(\text{unknot})$ is $\mathbb{Q}[x] \otimes \wedge(y)$ which is infinite dimensional. But we want to reduce this to a finite dimensional thing by getting rid of this, that is the goal.

[Sasha: The obvious thing is, does the same thing work, can you get those other homologies as specializations of this one?]

You have a complex with three differentials. If you take homology with respect first to $d_1 + d_2$

and then d_3 you get one theory, but if you take first with respect to d_2 and then d_3 you get a different one.

[Kwan: What is the intuitive way of thinking the $0, 1, 2$ lower indices?]

This is very complicated. There is a conjectural interpretation of H_2 with Floer homology, but that's probably not what you want.

[Tony: Do these have any extra structure?]

We want the functoriality of cobordism. There's $SU(n)$-equivariant versions, but there is no, say, multiplication. You have $H(L_1 \sqcup L_2) = H(L_1) \otimes H(L_2)$.