Graduate Topology Conference Notes
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1 Vaughan Jones, Planar algebras

Thanks for getting up an hour early. The title of the talk is supposed to be "Planar algebra" singular. The idea is that there's linear algebra, this is between linear and multilinear algebra.

I wanted to start with the subject of knot theory. I wanted to start nice and gently with Conway's tangles and the Alexander polynomial.

The original idea of Conway was that a tangle is a bit of a knot. A tangle is a piece, cut out by a circle (in projection). What we have left is a tangle. A tangle with $2n$ boundary points is called an n-tangle. Tangles are to be considered up to isotopy. That is, three dimensional isotopy inside the disk (the ball in three space).

Moving on to linear skein theory, and if you talk to Conway remember the "linear," we have W_k the vector space $\mathbb{C}[\text{all } k-\text{tangles}]$ under a quotient. This is infinite dimensional. We want something more manageable. We want to quotient out by the skein relation, $T_+ - T_- = zT_0$, where T_+ and T_- differ by a crossing and T_0 is the smoothing.

This is a finite dimensional vector space. You can also have little circles, but W_0 has dimension one. This means there's a polynomial invariant of links described by this, and that's the Alexander polynomial. He used this as a computational tool, and it turned out to be more effective than it seems.

It turned out that you can generalize this with three coefficients in here, and you get that the dimension is one. What is left is the HOMFLYPT polynomial, which has two variables. The final polynomial is harder to compute; starting with n crossings you have to deal with 2^n things at the end. It's not known that it's not polynomial time, but it's of a type that is not commonly thought to be.

There's one other thing, we're trying to work toward a calculus of tangles. Let me give another construction. We have the inner product on tangles. Suppose you have two elements of W_k, then all you do is join them up along the boundary. Then you eliminate the boundary disks and get a scalar. So you get $W_k \times W_k \to W_0$. Now you can start to ask more difficult questions. You could apply this in the case of the Alexander polynomial and ask about the rank. You can reduce the tangles down to $n!$, but that doesn't mean that the rank of the matrix is $n!$. So what is the rank of this matrix for given a, b, c? This has a lot in common with the skein relation. When you're a mathematician, you come across something like this and you want to know what the structure is. You have operations on tangles.

Excise all of the topology and you get a disk with a bunch of disks removed and strings joining the excised disks. Then you plug in tangles to get a tangle output. Both the inner product and the skein relation can be so expressed. This is an effective computation strategy. In practice it should not be hard to compute the HOMFLYPT up to 50 crossings or one specialization up to 200.

I have not discussed orientation. Everything should have had an orientation. To handle things we need inputs and outputs, and need to distinguish them. There need to be as many outs as ins. I didn't say, in order to make anything work you have to have a first string. So you also need to tie down the cyclic order on the boundary. You could split into two pieces, there's another nice operation on tangles, which is concatenation. If you have two tangles you can get the product tangle T_1T_2 and the vector spaces W_k are algebras. If you take the one obvious way of doing this, with all of the top ones going out and the bottom ones going in, you find out that the HOMFLYPT algebra is a well-known object, a (the) Hecke algebra. Its dimension is $n!$ and you can find the usual presentation by putting in the crossings. If you change some of the orientations I don't know the algebra structure.

So this involved throwing away all of the topology. If we just consider the disk with some orientations on the boundary, some of these choices might be better than others, you want a lot of patterns in the middle. The worst possible choice with no crossings is when you have all the outs then all the ins. The best choice is when they alternate. The number of ways is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. That worst one concatenates trivially so it's the identity.

So whatever the algebraic structure is, it has a very rich identity, all of the Catalan things are the identity. So I claim that this choice of orientation is a good choice, it lets you do a checkerboard shading. I'm going to copy the tangle I've drawn above, it's hard on the fly to get things to be even, the probability of a random integer to be even rapidly approaches zero. Shade the regions where the boundary goes clockwise. This takes us to the point where I'd call this a planar tangle. You need the disks, the strings, and the orientations (or shadings). The algebraic structure that is relevant is an operad, I'm not going to say what operads are, with a little more. You can look at the uncrossed tangles, which are called Temperley-Lieb. We already have an interesting problem, what is the rank of the inner product on this part of it? We also say that disjoint unknots should be δ. This matrix will have entries powers of δ.

I'm running out of time, let me leave topology. Now we're in the plane with these operations. Maybe we can put other things in the disks besides crossings. There's one thing we can put in there, I'm not going to remember it exactly right in my jet-lagged state, but if all the internal disks have four boundary points, then we can put "things" in instead of crossings, and thot thing we put in is $T_\infty - aT_0$, where these are the two possible smoothings. What comes out in the case of a knot, I don't have to worry, now, about over and under crossings, well, you have to shade it, then you get an underlying graph, each of the shaded regions is a vertex on the groph, and the edges will be the crossings, the doblu points in the projection. I can plug my "thing" into all of these. Then I can do my calculation and end up with something in W_0 getting a function of δ. This turns out to be the chromatic polynomial of this graph. This is merely a polynomial in δ^2 which, evaluated an n gives the number of ways of coloring the graph with n colors.

So now we have some reason to stick in things other than crossings. It's risky to say in public, I haven't investigated this, but suppose your internal disks didn't happen to have, just two things, you had many things, you can still naturally generalize the "thing" to put bigger copies of the thing in the appropriate places. It's not clear what it means, this should be positive at $n = \delta^2 = 4$. You need this dual basis for Temporley Lieb.

Now I think at this stage the definition should be clear. Planar algebra is a sequence W_k of vector spaces so that the elements can be inserted into the internal disks of planar tangles in a natural way. I've done enough examples. That's a planar algebra. We've seen that the skein theory gives one and we can get the chromatic polynomial from the Temporley Lieb algebra as well. That's the definition, and the incredible thing is that we got an incredibly rich set of objects. In terms of credits, this was arrived at by myself for considerations in functional analysis. Other people got here other ways. Greg Cooperberg got here in some way. This particular approach suggests divide and conquer. This is like Temporley Lieb, due to Busch and myself. In Temporley-Lieb W_k was the span of things with no crossings. We said to ourselves, we'll double the number of boundary points. If the tangle we deal with has six then we double each one. We consider ways of joining red to red and blue to blue. This can be like a doubled Temporley-Lieb thing, but doesn't have to be. So the linear span of these is W_k and we can reduce the one dimensional thing not to one dimension but to two.

The dimension in this case is the next of some series of Catalan numbers $\frac{1}{2k+1}\binom{3k}{k}$. These are Fuss-Catalan algebras, and seem to have nothing to do with topology or knot theory, but they have associated statistical mechanical importance.

Let me finish off by listing other examples.

- Tensors, if you have a tensor T_1 any time you have some indices, you can plug in and get an answer out. So you can contract over edges in you operation. Anothere thing to do is the shading, you can put the indices in the shading. Sum over the internal ones, this corresponds in statistical mechanics to vertex and spin models. Once you have this you can get all kinds of other planar algebras.
- finite groups
- homogeneous spaces of finite groups
- representations of compact groups
- finitely generated discrete groups All of these embed in the theory of planar algebras.
- quantum groups

I could go on but let me end this by saying all of these are special cases of subfactors, described by Von Neumann and someone else in the 30s.

[Let's thank the speaker.]

2 Computing stable homotopy groups using number theory Andrew Salch, University of Rochester

I lost my name tag, but, I'm Andrew. I want to start by defining stable homotopy. When I write π_* I mean stable homotopy. So $\pi_k^{stab}(X) \overset{\lim}{=\!\!\longrightarrow} \pi_{n+k}(\sigma^k X)$.

So we have a classical Adams spectral sequence with E_2 term $Ext(\pi_p, \pi_p)$ which goes to $\pi_(S^p)$. So with mod p Eilenberg MacLane spectra you can generalize and get $\mathbb{F}_p$ as the homotopy of a point. So you can replace this with E and get $\pi_*(S_E^v)$.

It's easier to take things in terms of comodules, just technically. So what theory E do you plug in to get the nicest equations, to get not too many differentials running around but still get the p-local theory. So it turns out you want to use BP. Don't worry if you don't know about this.

Classically you can do this at the prime two, maybe with a computer out to the 200th term or something, but in this BP sequence it was calculated into the thousands for the prime three by hand. So BP_* is $\mathbb{Z}_p[v_1, v_2, \ldots]$ and BP_*BP is $BP_*[t_1, \ldots]$.

To understand why these are interesting you have to go back to Quillen. A formal group law over R is a $F \in R[[x, y]]$, which are useful for cyclotomic extensions. These have to satisfy $F(x, y) = F(y, x)$, $F(x, F(y, z)) = F(F(x, y), z)$ and $F(0, x) = x$. This is a group with no points. What is a morphism of these things? Suppose you have F, G formal groups over R. This is a power series in one variable, $f \in R[[x]]$ with $f(F(x, y)) = G(f(x), f(y))$. I haven't given you enough to build the category, but one can.

This category has universal objects, let me tell you about them. There are two functors FGL which takes R in the category of rings to the set of formal group laws. Then SI takes R to the set of strict isomorphisms. You say that a morphism $f = \alpha x + \ldots$ is strict if α is one. So the set of strict isomorphisms of FGLs over R. So this gives you a groupoid, and the entire structure (FGL, SI) goes from rings to groupoids. This is corepresentable by MU_*, MU_*MU. You can go from a cohoomlogy theory with a complex orientation to a formal group law. This turns out to be a universal formal group law. That turns out why MU is really powerful.

Now BP is a retract of MU localized at a prime. It turns out that BP is the universal complex oriented cohomology theory over a $Z_(p)$-algpbra. So (BP_*, BP_*BP) has the same formal property. Maybe I've convinced you now that this is a powerful theory. That's kind of why it's big and interesting. Now how do you get to the E_2 term of this thing? That hasn't been computed. The most effective way of computing this we have so far is to use the chromatic spectral sequence. Once you know E_1 of that, E_∞ of that is E_2 of the sequence we're looking for.

To get to that you need cohomologies of certain groups and then you you need a bunch more spectral sequences. What I really want to talk about today are these groups, called Morava stabilizer groups. There's a number theoretic invariant of formal group laws called their height. You can define $K(n)_*$ whichi is $\mathbb{F}_p[v_n^{\pm 1}]$. So v_n captures height n cohomology theory.

The simplest one is height infinity, $H^*(\quad, \mathbb{F}_p)$. The most famous height one theory is KU, the most famous height two is tmf. No one knows what to expect about the higher height theories. Let me tell you about the formal group law over KM_*. So F_n is the Honda height. You take $\log_{F_n} = \sum_{i>0} x^{p^{in}}/p^i$ and $Fn(X,Y)$ is $\log_{F_n}^{-1}(\log_{F_n}(X) + \log_{F_n}(Y))$. So F_n over $K(n)_*$ in $K(n)$ as the formal group law.

So to compute these you do it by localizing over a prime; we also want to localize at a height. The chromatic spectral sequence will help patch together the height local information. The nth Morava stabilizer are the strict automorphisms, well, let $D_{1/n,K}$ be the Brauer invariant $1/n$ division algebra of center K. For the stabilizer you let K be $\mathbb{Q}_p$ but you can generalize this with a finite extension of $\mathbb{Q}_p$. You can talk about the maximal order (compact subring) of $D_{1/n,K}$ will be written $\mathscr{O}_{1/n,K}$. Then $1 \to \$_{1/n,\mathbb{Q}_p} \to (\mathscr{O}_{1/n,\mathbb{Q}_p})^\times \to (\mathbb{F}_{p^n})^\times \to 1$ defines \$.

Let me talk about the Brauer invariant $Br(K)$ is $H^2(Gal(K^{sep}), (K^{sep})^\times)$ So when K is a local field $Br(K) = \mathbb{Q}/\mathbb{Z}$

You can embed any defree n extension L over K in $D_{1/n,K}$. There's another way to do this, the useful way to do this for stable homotopy, you get an m extension and you get this kind of diagram:

$$\begin{tikzcd}
D_{m/n,L} \arrow[dr] & \\
L \arrow[u] \arrow[r, dotted] & D_{1/n,K} \\
K \arrow[u] \arrow[ur] &
\end{tikzcd}$$

[An example for the prime five.]

This hasn't been worked out above three. We can see $H^*(\$_{1/(p-1),\mathbb{Q}_p}, \mathbb{F}_p \to \prod H^*(C(\mathbb{Z}/p), \mathbb{F}_p)$.

Jesus, I'm just going to stop.

[Our time is up, let's thank the speaker.]

3 A new construction of simply connected spin-6 manifolds
Ahmet Beyaz, UC Irvine

In this talk I will talk about compact simply-connected (therefore oriented) smooth torsion-free six manifold with the third Betti number $b_3 = 0$. It is spin, meaning $w_2 \equiv 0$.

There is a classical theorem describing these, from 1966

Theorem 1 *(Wall)*
The diffeomorphism type of M is given by a free Abelian group $H = H^2(M)$, a symmetric trilinear form $\mu : H \times H \times H \to \mathbb{Z}$, and the first Pontryagin class $p_1 : H \to \mathbb{Z}$ satisfying $\mu(x, x, y) \equiv \mu(x, y, y) \mod 2$ and $p_1(M)X = 4X^3 \mod 24$.

An example is $H\mathbb{CP}^3$. These are homotopy equivalent to $\mathbb{CP}^3$ and they have the cohomology ring $H^*(\mathbb{CP}^3)$, so $H^2(H\mathbb{CP}^3) = \langle x\rangle$ and $H^4 = \langle x^2\rangle$, the top and bottom are $\mathbb{Z}$ and the others are zero. There are $\mathbb{Z}$ of $H\mathbb{CP}^3$. I will construct these first and then go to the general case. For these note that $b_2 = 1$.

Note that H^2 has only the generator x and note that $x^3 = 1$ and $p_1(M)X = 24k + 4$. I have an associated four-manifold X^4 with $\sigma(X) = 8k + 1$ and $\alpha \in H^2(X, \mathbb{Z})$ with $\alpha^2 = 1$. This α should be characteristic primitive. This means $\alpha\beta \equiv \beta^2 \mod 2$ and primitive means that if you take α out of the lattice then the dimension drops by one.

Now I'm taking the two-disk bundle B^2 over X and I will denote it M_α with Euler class α. Here $\delta M_\alpha = \#_{b_2(X)-1} S^2 \times S^3$. This is by Duan and Lian, 2005. Now I construct M by capping off the boundary. I write it as M_α connected sum (over the second Betti number of X) with $B^3 \times S^3$. Tracing the cohomology classes gives us the Gysin sequence and $b_2(M) = 1$. Say $H^2(M) = \langle x\rangle$.

Say that $i : X \hookrightarrow M$ and I claim $i^*X = \alpha$. This will give me the opportunity to calculate the intersection form on M. I also know that $PDx = iX$. So $X^3 = PD(x)^3 = X \pitchfork X \pitchfork X$. The first intersection is a surface representing α, and the triple is α^2 points. So $X^3 = \alpha^2$.

Secondly I'll see the Pontryagin class $p_1(M) = p_1(X) + \alpha \cup \alpha +$ torsion. Since the manifold is torsion free we have just the first two terms. The normal neighborhood of X has Euler class α and then [unintelligible]. So $p_1(M)X = p_1(M)PDx = p_1(X)[X] + \alpha \cup \alpha[X] = 3\sigma(X) + \alpha^2$ by the Hirzebruch signature theorem. Because of the initial data here I have $24k + 4$. So I constructed the manifold corresponding to k.

[Why are these spin?]

There is only one class and the intersection of it with itself is always intersection with, it's trivial, wait.

$w_2(X) = (w_2(X) + \alpha)x \equiv 0 \mod 2$.

Now I will give the case for $b_2 = 2$. I don't have time for $b_2 = n$. In this case, I have the Wall theorem, $H^2(M)$ generated by x_1, x_2 and the symmetric trilinear form has $x_1^3, x_2^3, x_1 x_2^2, x_1^2 x_2$, and $p_1(M)$. My building blocks in this case are $X_1,\ X_2$ and $\alpha_{ii} \in H^2(X_1)$ characteristic and α_{ij} for $i \neq j$ primitive and represented by spheres in four-manifolds. I also need an intersection condition $\alpha_{ij}^2 = \alpha_{jj}\alpha ji$. We need to show that such data exist (they do) and the construction gives me, I also need a condition an the signatures, $\sigma(X_i) = 8k + \alpha_{ii}^2$.

I have α_{ii} in the disk bundle M_{α_i} and α_{ij} which can be traced to the boundary. I take these, represented by embedded spheres smoothly, and trace them into M_{α_i}, taking the spheres out and gluing over them. Then I connect and cap off the baundaries and the construction eventually yields $x_i^3 = \alpha_{ii}^2$ and $x_i^2 x_j = \alpha_{ii}\alpha_{ij}$. This is it, any questions?

[Have you ever thought, can these be constructed with algebraic geometry?]

Yeah, I wouldn't call that algebraic geometry, i'ts smooth toplogy. There are two references in the paper. The interesting one says they can be constructed by knot surgery in S^6.

[How about $H\mathbb{CP}^k$ for higher k?]

I did not think about it.