Graduate Topology Conference Notes
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1 Why is Khovanov Homology so Cool
Allison Henrich, Dartmouth

(A basic introduction to Khovanov Homology, based in Viro)

2 The Torsions of Reidemeister, Milnor, and Turaev
Chris Truman, Maryland

After this talk we have a tea upstairs. I'm definitely going to define Reidemeister torsion, we'll see how far I get. References are Turaev, "Intro to Combinatorial Torsions" and "Torsions of 3-dimensional manifolds." This talk is shamelessly plagiarized. In 1935 this was introduced to classify lens spaces. Franz generalized this to arbitrary dimensions. A lot of interesting stuff happened that I'm not going to talk about, but the next part of the historical narrative I'm going to take up is Milnor in 1965, who noticed the Alexander polynomial of a link in S^3 can be thought of as a torsion of the link exterior. That's what we now call the Milnor torsion.

In 1976 Turaev introduced the maximal Abelian torsion, now called Turaev torsion. So, in the 80s, Turaev introduced refinements to torsion and a little later, there's some classical indeterminancies that Turaev showed how to resolve. In 1996 Meng and Taube used the refined torsion to compute the Alexander polynomial from the Seiberg-Witten invariant in dimension three. The torsion is combinatorial, so that's convenient. This doesn't work in dimension four. It also shows up as an Euler characteristic in Ozsvath-Szabo homology, which sounds kind of cool.

First I'm going to do some notation, in purely linear algebra mode. Let V be a finite dimensional vector space over $\mathbb{F}$ and a, b two bases. Then $[a/b]$ is the determinant of the change of basis matrix from b to a. So $[a/b]$ is a unit, $[a/b][b/a] = 1$, $[a/a] = 1$ and $[a/b][b/c] = [a/c]$. I'll generalize this to a chain complex, I'm going to do an alternating product.

So suppose I have (C_*, c_*) is a based acyclic finite complex of finite dimensional vector spaces over $\mathbb{F}$. Based means that for each i I have c_i a distinguished basis of C_i. Acyclic means the homology is zero, the sequence is exact. All of these are finite dimensional vector spaces, so, I'm going to compute $\tau(C_*, c_*)$ in $\mathbb{F}^\times$. Choose a sequence $b_i \in c_i$ for each i so that δb_i is a basis for $im(\delta_i)$ in c_{i-1}. If I look in dimension i the images of the $i+1$, they will be a basis for the kernel. So $(\delta_{i+1} b_{i+1})(b_i)$ is a new basis. So I define $\tau(C_*, c_*) = \left[\frac{(\delta_{i+1} b_{i+1})(b_i)}{c_i}\right]^{(-1)^{i+1}}$. You can do this when things aren't over a field, and you can do it for nonacyclic complexes. For the first you use K_1 instead of using the determinant, that's for Whitehead torsion.

I want a topological invariant. I want a based acyclic finite complex over a field, associated to X. For Reidemeister torsion you look at the twisted homology.

Let X be a finite connected CW-complex and let $\hat{X}$ be the maximal Abelian cover of X, meaning it corresponds to the commutator subgroup, having deck group $H_1(X)$ which I will call H. This makes things simpler, the calculations. I'm going to put a $*$ here because I'm going to do something that will come back and bite me later. I'm going to choose a lift of each cell in X to $\hat{X}$ and I'll randomly orient and order those cells. Now I have a basis for the H-action on $\hat{X}$.

This will turn into my distinguished basis in a little bit. This makes the cellular chain complex of $\hat{X}$ into a free $\mathbb{Z}H$-complex. So if I have a homomorphism $\phi : \mathbb{Z}\mathbb{H} \to \mathbb{F}$, I can construct the ϕ-twisted complex $C_*^\phi(X) = \mathbb{F} \otimes_\phi C_*(\hat{X})$. This has a distinguished basis coming from my choice of basis. I don't know if this is acyclic. I can define $\tau^\phi(X)$ as $\tau(C_*^\phi(X))$, basis if acyclic, zero otherwise.

So we mod out by something to get $\tau^\phi(X) \in \mathbb{F}/\pm\phi(H)$.

There's a natural map $H_1(X) \to H_1(X)/Tor(H_1(X))$ which induces a natural map on the group rings. The second one here is $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_{rk\ H_1}^{\pm 1}] \hookrightarrow \mathbb{Q}(t_1, \ldots, t_{rk\ H_1})$. So this composition is μ and $\tau^\mu(X)$ is the Milnor torsion.

Turaev torsion improves on Milnor torsion by taking the torsion part into account. Look at $Q(H_1(X))$, the classical ring of quotients of the group ring $\mathbb{Z}[H_1(X)]$, localizing at all non-zero divisors. So I have a sort of natural inclusion of this group ring into this localization. "It turns out," meaning I don't want to prove it, that $Q(H_1(X))$ is a direct sum of fields, which are $C_i(t_1, \ldots, t_{rk\ H_1})$ where the C_i come from the representation theory of the torsion of H_1; one is $\mathbb{Q}$ which corresponds to the trivial representation, giving the Milnor torsion. So $\mathbb{Z}[H_1(X)] \to \mathbb{F}_i$ by one component of the composition of inclusion with the unique splitting isomorphism, and $\tau(X)$ is $\phi^{-1}(\oplus_i \tau^{\varphi_i}(X))$.

[What are the interesting applications?]

You can compute Seiberg-Witten invariants, and Casson invariants except where they were originally defined.

[Could you say something philosophical about the relation among these?]

Reidemeister is wide-open, choose any homomorphism. For Milnor torsion it's a natural choice, but it discards information. Turaev squeezes a little more information with a lot more work.

[What sort of things can you distinguish between for a knot complement?]

It's exactly the Alexander polynomial in that case, the Turaev torsion is the Milnor torsion since homology is free.

3 Representing Homology Classes with embedded submanifolds Zuzsanna Dancso, Rutgers

4 Intrinsically n-linked Complete Bipartite Graphs Danielle O'Donnol, UCLA