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Before I launch off into what I was going to talk about, the motivating problem for all of this, it's all algebra, it's understandable, I wanted to ask if you have any questions about anything up til now. Are you as smart as Margaret, in this poster?

[I have an issue, but I don't know what it is. Samir and I were trying to compute the cohomology of LS^2. I worked up to H^5.]

No, that's wrong. I think the equivariant one is one in every dimension, and the other one is two copies of $\mathbb{Q}$. It's a little matrix with two ones in a row, maybe, it's just linear algebra.

Ralph Cohen just wrote a paper computing this for $\mathbb{CP}^n$.

Here you have the cohomology of the manifold, and here you have the cohomology of the based loop space; then the tensor product is a model for the free loop space. Just for cohomology, doing S^2 you have two groups for S^2 and then the loop space, you just take the generators and shift them down by one, getting two elements, in degrees one and two, and then take the free algebra generated by that, and you get a $\mathbb{Q}$ in every dimension. The differential of the E_2 term is over two and down one. So you can write down an exact sequence, called the Wang sequence, and you see that the Betti number is at most two.

If you know the homotopy groups of an H-space, you take the dual and take the free algebra, and then there's a secret differential that the space puts there. Before that there was the Leray-Serre spectral sequence which gave you some help, even if it was hard to get answers.

The model gives you an actual formula so that you can do it, but this gives you something to check your work, another way of looking at it.

I wanted to, well, whenever you do this thing here with cohomology, you get homotopy shifted down by one, and then the model has differential zero, which follows from the fact that the cohomology is a free graded commutative algebra. That follows from a Hopf algebra over the rationals with a $\mathbb{Q}$ in degree zero being a free algebra. The first construction of the model, you're done at the first step.

These models in this mathematics is having a rebirth due to string theory. I tried to explain some points in my lecture at Columbia. The idea that these models, just algebraically, can be related to these trees, that's new information from the nineties. The fact that it has this picture, that's relevant because quantum theory and string theory have such pictures.

What I was going to do was lead up to one of the computations for the quantum theory. This is natural in context of this stuff from the seventies. In the late sixties we were trying to classify manifolds. We had no hope to classify three and four manifolds, but we were trying to do higher ones. Poincaré, they had figured out surfaces and then he introduced cell decompositions, the elements of chain complexes, and then talked about homology classes. He talked about the ranks of Abelian groups and torsion coefficients, and then he introduced the homology sphere, it's got the integral homology of the three-sphere, but it's not the three-sphere, it's not simply connected, and then he worked out a bunch of things, fundamental group, Morse theory of three manifolds, eventually formulating the Poincaré conjecture. He stated this as, a space with trivial fundamental group is simply connected. In our language that doesn't make sense.

So everyone thought that it would be harder to deal with higher manifolds, and then it was a shock when Thom showed that you could classify with cobordisms. Smale showed that in dimension six or higher the manifolds that are cobordisms are actually products so the manifolds are diffeomorphic. Up to torsion the Pontryagin classes and the [unintelligible]determine the diffeomorphism type of the manifold.

So you have (X, "p'') which must be enhanced by some torsion. It's like the Pontryagin class. Suppose you wanted to count whether there were an infinite number of manifolds in that homotopy type. So you can transform with $\sigma \in Aut\ X$ and get $(X, \sigma p)$. So you have to count the orbits of the p under self-homotopy equivalences. What can you ask about any group? For the torus, $Aut\ X$ is $GL(n, \mathbb{Z})$. The automorphisms of $K(\mathbb{Z}^n, k)$ is $GL(n, \mathbb{Z})$. So you have to put these together in a kind of a Postnikov system. A subgroup of a free group can be an infinitely generated free group, and now things start going to pieces. I'm like 29 years old, I don't know all these fields of math, but I had heard of an arithmetic group. So I said, maybe this thing is an arithmetic group. I went and asked Roger Howe, "what's an arithmetic group?" He said, it's the integer points of a $\mathbb{Q}$-algebraic group, that is $GL(n, \mathbb{Q})$. An algebraic group is a subgroup of matrices defined by algebraic conditions. If the defining conditions are defined over $\mathbb{Q}$ you would say it was over $\mathbb{Q}$. So it turns out that an arithmetic group is an algebraic group over $\mathbb{Q}$, and then look at the integer points, the elements which preserve the integer points. Then you have anything commensurable with that, meaning it differs by something of finite index or coindex.

I wanted to take the subgroups fixing an element or something, and then the conjecture was, well, $AutH^*(X)$ was an arithmetic group, so that $Aut(X)$ was too and the map $Aut(X) \to AutH^*(X)$ was induced by an algebraic group morphism. So I knew you could get rational homotopy. I had heard that Quillen had made an algebraic model of rational homotopy theory. It took me thirty years before I could write things down and calculate with it. You talk about coalgebras, not algebras if you're a good algebraist. So he had this theory, an algebraic model with large cardinality. You may have heard of Van Kampen's theorem. You can make a free group on the cells, and then you take a quotient. Kahn completed this to a semisimplicial picture, for the loops, and then there's a theorem, for a free group you can make it free nilpotent, there's a functor, you mod out by commutators of level $k+1$. This thing has roughly one generator for each cell. Now, nilpotent groups can be tensored with $\mathbb{Q}$. When you tensor them with $\mathbb{Q}$ they're isomorphic to their Lie algebras. Then a guy at MIT proved that for simply connected spaces, if you take k bigger than the dimension you lose no information. So you tensor with $\mathbb{Q}$ and pass to Lie algebras. You use the formula $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$. Notice when the group is nilpotent the expression terminates, which tells you how to go from multiplication to the Lie algebra. Then Quillen passes to a differential graded Lie algebra. Over the rationals passing to the differential graded Lie algebra is an equivalence. Then you go to coalgebras and then Hopf algebras.

This is one of these great papers. A good math paper can have a life of thirty years. Witten's papers, some of them have lives of twenty years. So I didn't like any of this. The word nilpotent used to make me nauseous. That's why I don't like words, if you don't know the definitions it makes you sick. So *dg* colgebras, those are kind of like *dgA*s. So if you have a coalgebra, you can get an algebra structure on the dual. If you have an algebra you don't necessarily get a coalgebra on the dual.

So where are we? They weren't equivalent to differential graded algebras, if these were finite you could get to differential graded algebras. Still, this is over $\mathbb{Q}$. So I asked myself, what does this have to do with differential forms? You have a natural algebra of forms. Those are normally defined over the reals and they have a very high infinite dimension.

What I'm going to say is slightly contradicted by Chen's thesis. This *dgA* of forms doesn't come from a coalgebra. But I said, let's find a finite type algebra that maps into here inducing an isomorphism on cohomology, and then maybe that will come from the Quillen theory. So I started trying to do this, and we built this thing, and then one can start asking, what does this have to do with the coalgebra. So to even ask the question I'd have to go back and learn all of the steps up to that point, but then I knew about the Postnikov system and also I noticed you can look in Whitney's book, the Postnikov system isn't made out of manifolds, but Whitney has forms on the simplicial complexes, they agree on the edges in the direction of the edges. If Quillen had known about this book he wouldn't have written this paper. Whitney's book already had this functor. All the experts accepted this paper, but in the textbook of 1957 already contained this functor. It's over $\mathbb{R}$ in Whitney's paper, but you can just modify it slightly to get it rationally. So you can put a $\mathbb{Q}$ in Whitney's book. So now you have rational forms and you can look at the Postnikov system and build a model, and have an alternative algebraic model constructed directly from the algebraic forms. In the eighties they showed these systems were equivalent, that didn't matter to me.

This was a digression, I was talking about the arithmetic group thing.

[You said that the motivation was to get a DGA, what about forms?]

I had this book in my hand, I was going out with Michael's mother, and after starting dinner I was reading it, and the phone rang, and I told someone I was looking at Whitney's book, and he said that there was nothing in there. The consensus was that it was about advanced calculus, in some eccentric way. I remember once as a graduate student I woke up in the middle of the night, and said "this famous topologist, this seminal mathematician, invented differential topology, Milnor was just bearing the fruit of it, and he wrote just this one book, what is in this book? There must be something good in there." So I ignored what this other mathematician thought, I went and looked in it, I'm reading after we ordered food, and you just read it, and then there's sort of a de Rham theorem, and it's like a one-line version that takes eighty pages in the Annals. The body of this paper is fundamental, but the motivation was answered by this one line construction.

In the nineties I was reading this, it's like a combinatorial construction, to think about quantum things.

[Isn't yours combinatorial?]

No, Whitney puts an infinite dimensional object on each cell. Quantum would be chunkier.

Okay, so getting back to what we were talking about, what is the nature of this map, I need this $\mathbb{Q}$-algebraic group, I'm going to tell you the answer, what $(AutX)_{\mathbb{Q}}$ is. All integer matrices for one lattice, they'll intersect in something of finite index. You can read about all of this in $InfinitessimalComputationsinTopology$. This will give you a differentiable manifold with finite ambiguity. It's around 1978. It's IHES number 47.

All right, take simply connected spaces, and I have the minimal model, the free one $\Lambda(--, d)$. This is well defined up to isomorphism by the homotopy type. It's not a unique isomorphism. This thing is equal to the automorphisms of the model; above the dimension of the space the construction is formal. So automorphisms of the truncated model modulo those homotopic to the identity.

Consider $i : \Lambda \to \Lambda$ which are derivations of degree -1. Then the subgroup quotiented out are those writable $exp(di + id)$. This would give you a chain homotopy to zero, so the exponent should give you a homotopy to the identity. The exponential is the ordinary exponential infinite series.

In this case if you think of the monomial filtration, you get all the guys and then all the guys that are sums of products, and then sums of cubics, and so on. d in the minimal model takes you down, and i preserves, so the formula takes you down, and in a given degree, you can only have monomials of a a fixed length, so in each degree you're finite.

This in fact defines a, what is the total group, it's a bunch of matrices with $Md = dM$. It maps a generator to another in that degree and it has to commute with d.

The claim is that the integral points of this are commensurable with $Aut\ X$.

If you try to do this with the Quillen algebras you have trouble because things are infinite. Then the map on cohomology is a map of algebraic groups. Then there's a theorem about them:

$$\begin{array}{c} G_{\mathbb{R}} \longrightarrow H_{\mathbb{R}} \\ \\ \Gamma_G \longrightarrow \Gamma_H \end{array}$$

The miracle is that if the map on top is surjective then the map along the bottom is quasisurjective, meaning the image has finite index. So you can whip things around. I'll tell you a beautiful theorem, two groups Γ, Γ' are commensurable, if Γ'' sits in both and is of finite index in each. Given discrete $\Gamma \leq G$ a Lie group, like with Γ in $PSL_2(\mathbb{R})$, this, among the discrete subgroups, say they're lattices, cocompact or [unintelligible]has finite volume, then Margulis showed that a discrete subgroup of a Lie group is an arithmetic group. There are only countably many arithmetic groups. Only countably many of the uncountably many discrete subgroups will be arithmetic. Any discrete subgroup is arithmetic if it's semisimple of rank at least two. If it has rank one, those are examples of negative curvature. If you go to hyperbolic three-space, there are only countably many. Except for this example there are only countably many. It turns out that for rank one, some groups are arithmetic and some are not.

There's a whole wold of such groups out there. He proved, notice that if you take $GL_n(\mathbb{Z})$ sitting in $GL_n(\mathbb{R})$. There are a lot of elements that commensurate this guy. Any rational matrix only has so many denominators, so conjugating by an integer matrix with the right coefficients will just move things around. There are no elements that take all of this to itself, but many elements, a dense set, commensurate these. A lattice in a Lie group is arithmetic if and only if the commensurator is dense.

So you do commensurability mathematics and you can take your understanding about rational theory and push it back to this theory, but you need to put the lattice in.

I was only going to take five minutes for that but I took almost all of the time. The appendix of my paper, you can work in the field without knowing the proofs with that appendix.

There's a famous problem that if you have a Riemann surface, compact, with arithmetic Fuchsian group, then eigenvalues of the Laplacian are bounded away from zero by something like $\frac{3}{16}$, or maybe $\frac{1}{4}$, or maybe $\frac{1}{2}$. I think it's known it's bounded away by 3/16. It's known it's bounded away. This conjecture is the Riemann hypothesis, think about that. The spectrum of the Laplacian sees the topology before it happens. This is a conjecture. Possibly the discrete spectrum is bounded away from zero in the noncompact case. Burton Randall would know it here. Peter Sarnak at NYU, ask him this question and you wouldn't be able to leav for forty-eight hours.

I only did all this because I wanted to have that *exp* of a derivation is an algebra map.

[Is the infinitessimal because of this *exp*?]

There are things about connections and exterior d. That's what it's about. Forms, that's a miracle, that the wedge product works. When you write down a differential form, like a symplectic form, the geometry of that form is complicated.

Okay, I think I'm just going to, the point is, there was this question but the point was to give a finite presentation of the subgroup being quotiented, the derivations that are of this form, commutators with d, give you the things homotopic to the identity. Therefore the homotopy classes of automorphisms are algebraic groups, and you have a chance to show this thing about arithmetic groups.

A lot of the people who work on these quantum things, they don't know about ∞ algebras. Time's running out, let me show you, I mumbled about constant terms. There's a difficulty with constant terms. Suppose I have a dgA with stuff in degree zero, I might have stuff on both sides of the origin, and sometimes some generators in degree zero as well. Then you have that.

Make a dgA, and there's a unit 1. Let me call all the generators V. To define a derivation you have to give a map $V \to V$ and then one $V \to V \otimes V$ and then one from V to $V^{\otimes n}$. This tells you what d of a generator is, $dx_j = \sum a_{ij}x_i + \sum a_{ijk}x_i \wedge x_k$. I left out the term $V \to k$. That's somehow the ground field. You can have a term like this, you could have generators in degree -1 which could be mapped up into degree zero. In the minimal models everything starts above $V \otimes V$. If you don't have the ground field you can eliminate the $V \to V$, this is the minimal model statement. So then you can squeeze the juice out. The algebraic process in quantum things, is, you have to do something to get rid of the constant term. Then you do the analogue of the minimal model and you get something with well-defined isomorphisms, which give you the analogues of quantum correlators. You get $d_1^2 = 0$ and then you take the homology. The new space will be the d_1 homology of V. So now you have $H \to H \otimes H$, and so on. On the dual space these are like correlations. This is a canonical vector space. In the original thing there were a bunch of choices. Quantum theory is like, you have something with a differential in it, you get rid of the constant term and then squeeze out the juice.

How do you get rid of the constant term, this is kind of like Terilla's talk, you have to solve a nonlinear equation. You need to look at the coalgebra side. You have coderivation generators $W^{\otimes n} \to W$. This is a linear functional that you can extend to a derivation, the dual of that is a map from k into W which is just an element. This extension is applying it to each factor, which is dual to, the free graded commutative algebra on W, where the generators go to $w \otimes 1 + 1 \otimes w$. You extend it to be multiplicative. The coalgebra acts on the generators primitively like this. This is a Hopf algebra. To be a coderivation means that if you apply the map and then do the comultiplication, it commutes as:

$$\begin{tikzcd}
\Lambda \arrow[r, "C"] \arrow[d, "D"'] \arrow[dr, phantom, "1\otimes D + D\otimes 1"] & \Lambda \otimes \Lambda \arrow[d] \\
\Lambda \arrow[r, "C"'] & \Lambda \otimes \Lambda
\end{tikzcd}$$

So usually you'll get a big algebra with an operator of square zero, we'll suppose it's a coderivation, which may have these terms like this, and then we want to get rid of the constant term. Getting rid of the constant term may be impossible and if possible may be non-unique. Each solution, we have this D here, with a d_0, d_1, et cetera. Each solution of the quantum master equation $De^x = 0$ gives a way to get rid of the constant term, and then I can squeeze the juice out and get the correlations. If D were a derivation, this would be $Dx(e^x) = 0$ so it would be $Dx = 0$. But it's a coderivation. Gdven an element x we can form, say $x \in V$ we can consider, extend it to a coderivation, mapping each monomial goes to $x\cdot$ This turns out to be a coderivation. What is *exp* of that operator? It's $1 + x + x^2/2 + \dots$ You just keep iterating these multiplications. That's what *exp* is. This is like creation. You just multiply. These are like particles, you multiply by x. This now is the dual, the exp of the coderivation preserves comultiplication. You compose $e^{-x}De^x = D_x$. This is an isomorphism, a comultiplication-preserving map. You get a new coderivation of square zero. It's a coderivation and $D_x^2 = 0$. You get $e^{-x}D_x^2e^x$.

How do you tell whether there's a constant term in the coderivation? If you have a constant term which is a then 1 goes to a, $D1 \neq 0$. So $D_x 1 = e^{-x}De^x = 0$. If you conjugate by that solution then you get rid of your constant term, you can do the compression. This is a feature that wasn't present in the homotopy system.

What if D is a BV operator, so that the deviation of being a derivation is a derivation in each argument, that's a second order derivation. If D satisfies $D^2 = 0$ and is a second order derivation then that deviation is a Lie algebra. They produced BV operators to quantize. In this case, $De^x = (Dx + \frac{1}{2}[x, x])e^x$. So in this case the master equation is $Dx + \frac{1}{2}[x, x]$. Solving this equation in the BV case is what the equation amounts to. So solving the quantum master equation amounts to telling which universe you're in, the initial conditions of the discussion. If you know your solution you can change variables and get $D(1) = 0$. That's the idea, and that's like pushing this algebra some more. The yoga is, solve the master equation to get rid of the constant term, now you can go to the minimal model of the discussion. That's the conjecturing. I'm going to go up to lunch. All of you should come up to lunch, especially those of you going to Park City. That's quantum math but they don't say the word quantum.