Dennis Seminar
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The first i such that $\pi_i(F) \neq 0$ gives an invariant of the bundle $E \to B$, in the obstructions $\theta \in H^{i+1}(B, \pi_i(F))$. example are, if you have an $\mathbb{R}^d$ bundle, you form the kth Stieffel bundle where the fiber is k-frames in $\mathbb{R}^d$ and the space is $d-k$-spherical, meaning $d-k-1$-connected.

This goes for $k = 1, \ldots, d$, and the obstructions, the $i+1$ goes $1 \leq i \leq d$. We get d invariants in $H^i(B, \pi_{i-1}(V_k))$.

I said last time the groups were cyclic or not. Let's go through the twisting in more detail. We can forget the first element in the frame to get an onto map $V_k \to V_{k-1}$ with fiber the sphere of dimension $d-k$. This is a geometric fibration. If you look at this sequence $\pi_{d-k}S^{d-k} \to \pi_{d-k}(V_k) \to \pi_{d-k}(V_{k-1})$. But V_{k-1} is $d-k+1$-spherical, so this group is zero. We have

$$\pi_{d-k+1}V_k \to \pi_{d-k+1}V_{k-1} \to \pi_{d-k}S^{d-k} \to \pi_{d-k}V^k \to 0.$$

We also have the onto map $\pi_{d-k+1}S^{d-k+1} \to \pi_{d-k+1}V_{k-1}$.

What's the composition $\pi_{d-k+1}S^{d-k+1} \to \pi_{d-k}S^{d-k}$.

All of these groups are cyclic. So it can either be infinite or finite. If it's finite then the one before it is infinite. If it's infinite the one before it will be (conjecturally) $\mathbb{Z}/2$.

Start with the generator of the fiber, that gives you the sphere of possibilities for a last element over a fixed $k-1$-frame, okay, I'm going to have to sit and think to make this computation.

You can reduce these modulo two to get the Stieffel Whitney invariants $\omega_i \in H^i(B, \mathbb{Z}/2)$.

V_d is the principal homogeneous space of a group, so π_0 is homeomorphic to a group, and there are two components, so this is $\mathbb{Z}/2$. Then the next group has to be a $\mathbb{Z}$.

The $\mathbb{Z}$ in even dimensions have no twisting. When we do the complex things, we only get the odd dimensional spheres, so you only get half the cases, and the appropriate groups will all be $\mathbb{Z}$. The first obstruction is $H^{2i}(B, \pi_{2i-1})$ and we will get d of these.

If you forget a complex structure then the c_i reduce to the appropriate ω_i. This means all the odd Stiefel-Whitney classes are zero for a complex manifold.

[Some quaternionic stuff]

First of all, to make this discussion, it's only about bundles, and you can talk about tangent bundles of manifolds, so the actual discussion about coordinates was unnecessary, and wrong.

There is no such thing as a quaternionic diffeomorphism. I'm actually, I've been through this on my own, when you write down a polynomial in z, it would be neat to have something like this in $\mathbb{R}^4$. The problem is when you write down a linear thing, bx, which side do you write it on?

There is something called a hyperK ahler manifold. You can ask for a Riemannian metric compatible with the complex structure, the metric gives you a canonical connection, you ask that it commute with J, that's called a K ahler metric. For a hyperK ahler manifold, you have a quaternionic structure.

These things are exotic. In the four cases you get Pontryagin classes, these were in the early forties, and the Chern classes were in the late forties, and the Stiefel-Whitney ones were in the thirties.

You can get a $\mathbb{C}^d$ bundle functorially by tensoring with $\mathbb{C}$. Let me make a remark, about complex conjugation, if you have a $\mathbb{C}^d$ bundle with J multiplication by i then you could pass to the conjugate bundle where J is $-i$. That multiplies the Chern classes by -1 in the odd dimensions. We define the Pontryagin classes of an $\mathbb{R}^d$ bundle are the (even) Chern classes of the complexified bundle. The odd ones could be written as Stiefel-Whitney classes. This is one of the reasons the Chern classes have taken prominence. From them you can reduce mod two and get Stiefel Whitney classes, and the Pontryagin classes come from them.

When the $\mathbb{R}^d$-bundle is orientable and d is even, you get an element in $H^{2k}(B, \mathbb{Z})$. This is orientable if the bundle is orientable. This $\mathbb{Z}$ is $\pi_{d-1}(\mathbb{Z})$. If d is odd then it's still orientable, the class is well defined, but it has order two. This is the only interesting new information. For a complex bundle, this is c_d. They have the same definition. That was around first, for the tangent bundle, if you have M^{2d} and you try to find a cross section, you get $H^{2d}(M, \tilde{Z})$. It was a theorem of Hopf or Poincaré that the obstruction vanishes if and only if the Euler characteristic is equal to zero.

There's a club of mathematicians who look for combinatorial definitions for the Pontryagin classes. Many of us wrote papers about combinatorial definitions for the Stiefel-Whitney classes, Stiefel, Cheeger, even I.

There's not even a natural relationship between combinatorial and complex structures.

These questions are a little sophisticated. Look at the c_i and p_i classes, tensor them with $\mathbb{R}$, then there is a geometric description of this, Chern-Weil. Chern considers Chern-Gauss-Benet the theorem of his life, a geometric description of the Euler thing.

For each tangent vector you know how to move the fiber infinitessimally, for each one you give an endomorphism of the vector space, like an infinitessimal. If you call that A it assigns to each tangent vector something. It's a matrix of one-forms. This is a one-form with values in matrices, then $dA + A \wedge A = \Omega$ is the curvature. Then Ω^i has a trace, which is the Chern class. The even ones are the Pontryagin classes.

One knows exactly, you need an invariant form of the coordinates, it's a miracle that things change by conjugation. If you have a metric, you can take the Pfaffian, the square root of the determinant, which leads to the Euler class. These expressions are important in other branches of math. Chern-Simons involves doing this for $d = 2$, and writing it as d of something. These are central, and pretty mysterious. They're not too mysterious, they're like first obstructions, but the entire role they play, whoops, I'm over.

Rene Thom did one more thing, you can apply this to a manifold by taking the tangent bundle. If you have a certain differential operator, you can compute things, the Atiyah Singer, Riemann Roch, which all compute the dimension of a set of solutions to some PDEs. You get formulae relating interesting integers relating geometric structures to these classes somehow evaluated over a manifold. Rene Thom showed, for a combinatorial manifold, the Pontryagin classes tensor $\mathbb{Q}$ exist. That's the context where the combinatorial formulae can be asked.

One other thing in 1957, this was Thom, in 1963 or 1964, Novikov showed that a homeomorphism carries the Pontryagin classes of one manifold to another.

In my thesis I studied the manifolds that are homotopy equivalent to one another. Basically it corresponds to the pontryagin classes. They basically vary freely. They're extremely significant, somehow, and still mysterious, if these didn't exist for general manifolds, I'd say they were the first obstruction, but tensored with $\mathbb{Q}$ they extend to this bigger realm of combinatorial structures.

These have to be better understood.