Dynamics Seminar
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Grafting coordinates for Teichmueller space

I'm going to talk about grafting and its relation to Teichmueller space, how it can give you coordinates.

I am going to start by talking about grafting. Let me say once and for all that S is a compact surface of genus $g > 1$ and $T(S)$ is the Teichmueller space of marked hyperbolic or sometimes complex or comformal structures (S is smooth). This is a $6g-6$ dimensional cell or a $3g-3$ himensional complex manifold. You start with some metric and you take γ an isotopy class of simple closed curves which we identify with the symple closed geodesics. We will cut it out and replace it with a cylinder which is the product of γ and an interval of length t. This has a well defined metric which is not fully hyperbolic. So this is $gr_{t\gamma} : T(S) \to T(S)$. Secretly I think of t as a weight assigned to a geodesic γ. So we allow γ to vary and we get a map gr : weighted simple closed curves $\times\, T(S) \to T(S)$.

Now a geodesic lamination is a foliation of a closed subset of X by complete simple hyperbolic geodesics. On a compact surface this is a strong condition.

There is a canonical way of identifying two for different metrics, it is just convienient to use the metric.

A transverse measure for a Λ which is a geodesic lamination assigns a Borel measure to each transverse arc $k : [a, b] \to S$ compatible with splitting and isotopy of transversals.

A measured geodesic lamination $\lambda = (\Lambda, \mu)$ is a geodesic lamination and a transverse measure of full support. This means we can't have spiraling geodesics.

This means if k is a transversal then the total measure of k is what we call $i(\lambda, k)$.

A sequence of measured laminations converge to λ if $i(\lambda_n, k) \to i(\lambda, k)$ for all compact k.

A few facts: $\mathscr{ML}$ (the set of measured geodesic laminations) $\cong \mathbb{R}^{6g-6}$ and finitely many tranversals $k_1, \ldots, k_N$ give you an embedding $\mathscr{ML} \hookrightarrow \mathbb{R}^N$.

What are you going to do with these things? Well, $t\gamma$ are dense in $\mathscr{ML}$. So it is the completion of the set of simple closed geodesics. You can approximate a sum of two with a single one.

Theorem 1 *(Thurston)*
Grafting has a (unique) continuous extension to $\mathscr{ML}$*, i.e.,* $gr : \mathscr{ML}(S) \times T(S) \to T(S)$.

The main theorem

Theorem 2 *(D, Wolf)*
For any $X \in T(S)$ *the map* $\lambda \mapsto gr_\lambda X$.

Let me just point out that $\mathscr{ML}(S) \to 0$, the empty lamination, is the origin for the cone structure on $\mathscr{ML}$. There is an action of $\mathbb{R}^+$ where you scale the measure.

This makes $\mathscr{ML}$ a cone centered at this point you've chosen.

Clearly $gr_0(X) = X$.

You can have a single lamination that admits many coordinate systems.

I want to discuss some of the ideas that go into this theorem, but I want to discuss a cousin, the Scannell-Wolf theorem.

Theorem 3 *For a fixed* λ*, the map* gr_λ *is a real analytic diffeomorphism.*

These both have something in common as far as overall strategy.

Local injectivity implies homeomorphism. This is because grafting is proper, so tha