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Gabriel C. Drummond-Cole

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I'll continue and we'll see later about the presentation. There will be a class this Friday. We have this result that if you have a $K(A, n)$ space then the different maps, the homotopy classes of maps into here are in one to one correspondence with $H^{n}(X, A)$. So if you have any space $Y$ you can take the point mapping into $Y$ and make this into a fibration. You take all paths in $Y$ starting at the given point, call that $\mathscr{P}$. Then evaluate the paths at their endpoints and that gives you a fibration over $Y$. The fiber over each point consists of paths from the given base point to that point. This is a fibration. The fiber over the base point is a natural group space. You think of the maps as being from the positive reals that are eventually constant. Then you compose by sticking one in at the endpoint of the previous ones. With this modification, you don't have inverses but you get an associative semigroup. This path fibration has the property that all of the other fibers aren't groups but they're acted on by this group, or object, the fiber over the base point. So this is what's called a principal fibration, where the fiber is acted on by a group. This is like a homotopy principal fibration. You can do this to $K(A, n)$. The loop space shifts homotopy groups. A map of $n$-spheres into the space is the same as a loop of maps of $n-1$-spheres. So the fiber is $K(A, n-1)$. This shows that all the $K(A, n)$ are group spaces, for $A$ Abelian. A group space or a loop space. This natural fibration has this group acting on the fibers.

There's a general argument by easy obstruction theory. Suppose you have a contractible total space and a fiber acts on the total space. Then any other fibration by the same group acting in this nice way on its fibers, you can look for an induced map


You can build this inductively using naive obstruction theory. To put something in a simplex the obstruction is in a contractible space. I start with, I'm proving that the category of $\mathscr{G}$-principal fibrations has a universal object. The space of all maps, this is a big space, it makes a contractible space.

So I just built this in the case where if I knew that $K(A, n)$ was a group, I guessed it was $K(A, n+1)$. In general, if you have a group, you just have to do it. So $\mathscr{G}$ is essentially $\Omega B$.

All right, so another way to state this theorem here is that the principal bundles over $X$ with group $K(A, n-1)$ are in correspondence with homotopy classes of maps $[X, K(A, n)]$ which are equivalent to $H^{n}(X, A)$. All of the questions about groups versus semigroups were answered by Milnor, he wasn't satisfied with this fuzziness.

These exist and are unique up to homotopy. This is a stronger statement, the space of classifying maps is contractible.

A $K(A, n-1)$ space is associated with a cohomology class $K(A, n)$ That is the first obstruction to cross-section. The $A$ is $\pi_{n-1}(F)$. There is a one to one correspondence between cohomology and bundles withh fiber $K(A, n-1)$.

You can have fibrations like this which are not principal. I was saying there was an action. Then the obstruction would be in twisted coefficients. You would get an automorphism of the fiber as you moved things around. General fibrations are organized by twisted cohomology. I am going to suppress the complete discussion of that more general case. Ordinary cohomology corresponds to principal fibrations, where $\pi_{1}$ of the base acts trivially on $\pi_{n-1}$ of the fiber. If you didn't know it was principal, then knowing this would tell you that the twisted thing would have ordinary coefficients. So you can pull back the first obstruction zero, and so since the one is principal, the induced one is also principal.

Let's study spaces with two nonzero homotopy groups. I've got this space $X(*, *)$. It's got these two homotopy groups in the two dimensions. You can go to the first nonzero homotopy group. For any space, you've heard of a few things in topology. You can take the space, how can I make a fibration over it. I want to construct a fibration with it as a base. Take paths. Then the first obstruction to cross section is in $H^{n}\left(X, \pi_{n-1}(\Omega X)\right)$. So there is a canonical element of $H^{n}\left(X, \pi_{n}(X)\right)$. Well, cohomology can be evaluatod on homology, so this is $H_{n}(X) \rightarrow \pi_{n}(X)$. This is the inverse to the Hurewicz homomorphism.

This doesn't work when $\pi_{1}$ is acting on itself nontrivially. This would be $\widetilde{\pi_{1}(X)}$ if $\pi_{1}$ were nonAbelian. So given any space you can take this canonical cohomology class, $X \rightarrow$ $K\left(n_{1}, \pi_{n_{1}}(X)\right)$. If $n_{1}$ is 1 I want $\pi_{n}$ Abelian. You are adding cells to kill all the higher homotopy groups. This is an isomorphism on the first group, so by the exact sequence, the other nonzero group goes to the fiber, so the fiber will be $K\left(n_{2}, \pi_{n_{2}}(X)\right)$.

A space is called simple if it is simply connected or $\pi_{1}$ acts trivially on all $\pi_{n}$ (including itself so $\pi_{1}$ is Abelian). Try as an exercise, $H$-spaces are simple. This is a space where you have a multiplication with a unit, so $X \times p t$ to $X$ is a homotopy equivalence to the identity. If you have two loops, you can draw them on the torus, and then extend along the product using the multiplication. But $\pi_{1}$ of the torus commutes. That's the idea of using this. By the way, I'm expecting some homework. This works for any space. You can pick off the lower homotopy group. So then $X$ is determined by the pair $\pi_{n_{1}}(X), \pi_{n_{2}}(X)$ and an element $k$ in $H^{n_{1}+1}\left(K_{1}, \pi_{n_{2}}\right)$. This works with any space by induction. Split off the first homotopy group and build such a map, and then take the fiber, we could repeat this but that's not what I
want. We have this map and the fiber. Where's the first obstruction to cross section. It's in the first nonzero group, so we can build just a two-stage system using this construction. Then there's a map into this fibration from the fibration of $X$ over 1 which is an isomorphism on the first two homotopy groups. Then I can lift this map as this thing is already defined on the "subset" $X$ so the obstruction to raising it is zero. This will be an isomorphism on the next groups. So I go on to build a map to $(1,2,3)$ and so on. This gives me the Postnikov system. What is this good for? You never get there, most spaces have an infinite number of homotopy groups. These things depend on $X$. Eventually you get out of the cohomology of any space $K$ so you can lift back once you pass out of the dimension of $X$. So mapping finite spaces into $X$ can be described by the Postnikov system.

This is all done with external algebra related to $X$. Something that I was involved with, started by Serre and Quillen, is that, well, take an Abelian group. and tensor it with $\mathbb{Q}$. You can kill all the torsion. YOu can talk about tensoring a space with $\mathbb{Q}$ or the $p$-adic integers or whatever you like. You can tensor it with $\mathbb{Z}[1 / 2]$. If the fundamental group gets in the way this doesn't work so well. We don't know how to tensor an arbitrary abstract group with $\mathbb{Q}$. This is all a consequence of the obstruction theory to cross section.

I was going to do an example, you might say, well, suppose we wanted to classify spaces, what would we have to know? We'd have to know the cohomology of these building blocks and we'd have to know, if inductively, if we know some cohomology we want to knwo how to build more. This is the Serre spectral sequence for a fibration. This is always smaller than the product.

Unfortunately the cohomology of the building blocks is complicated. The spectral sequence is brought to you by the space $X$, so it's complicated as well. They use a language, things getting killed, survivors. Anyway, in practice, I don't know, you have to be smart to use this. You can't just handle this. However, we found in the 70s, I did, using Quillen's work, found that there are nicet models. The first two examples, $K(\mathbb{Z}, 1)$ has comohology $\wedge\left(u_{1}\right)$. The next one is $\wedge\left(u_{2}\right)$. Rationally this continues to be true. If it's odd it's an exteriour algebra, if it's even a polynomial algebra. Now you're getting rational homotopy theory.

What one learns is, you don't take cohomology to study spaces. You build an algebraic model for the chain version. You learn that there are models with differentials. This is a model of the rational homotopy type of the space up to this point. Then you are going to model the fiber by taking the free commutative algebra, you take the one for the base and you just throw in the cohomology of the fiber. No matter what, you throw in the free algebra on the appropriate generators. You have a differential which describes the twisting, involving a formula in the $u$ s in the kernel of $d$. You could have like $u_{1} u_{2}+u_{3}^{2}$. Then you add $y$ in the fiber, with $d y$ equal to this. This extends the notion to $d$ because you chose this thing to be in the cohomology. So you add new variables to describe the next ones and so on. This gives you a model for what's going on. There's a miracle that makes things work which is that differential forms work. The abstract thing maps into the differential forms, and you can build them up.

If you build these in the minimal form, these things are rationally homotopy equivalent if and only if the things you construct are, if this is an algebra isomorphism.

Then these things are the homotopy groups, and you can turn this thing around. This is like a proof, the theorem, you have $Q$-differential forms and you inductively build a free differential algebra which maps down onto this and induces an equivalence. For the two sphere, you take the generator $x$ in degree two, with $d x=0$. you then get $x^{n}$, all of which are zero in $S^{2}$. So $d y=x^{2}$. So then you have $x y$ in degree 5 and $y x^{2}$ in seven and so on with $y^{2}$ in degree 6 . So now you have an isomorphism on homology. So you get $\pi_{2}^{\mathbb{Q}}\left(S^{2}\right)=\mathbb{Q}, \pi_{3}^{\mathbb{Q}}\left(S^{2}\right)=\mathbb{Q}$ and all the higher ones are zero. This theory is due to Quillen, it's still a fresh paper. Really good papers last thirty years. All the stuff in this about algebras, loop spaces, classifying spaces, it's still good today. This work of mine is a little later, what does this have to do with differential forms? He didn't consider this model because he's smart. He had coalgebras, Lie algebras, Hopf algebras, a differential coalgebra sounded like a differential algebra. So first you have to make forms over $\mathbb{Q}$, then you, it also gives you forms on all spaces. Then you can apply them in the Postnikov system to get models. These are "Rational Homotopy Theory" and "Infinitessimal Computations in Topology." I called it that because there was no analysis in topology, this is something with monodromy and forms. Connections are like infinitessimal descriptions of $\pi_{1}$. This was used to study smooth manifolds up to diffeomorphism. If you start computing examples, you get into a group theory problem. If you have a homotopy type, you have to look at how the automorphism of the homotopy type acts on Aut $X$. In 1970 nothing was known about these things. I wanted to do this for manifolds. Already when $X$ is the $n$-torus you get $G l(n, \mathbb{Z})$. These are arithmetic groups. I went to Roger Howe, I asked him, what's an arithmetic group. He said, you're kidding, how could you not know, it's the set of integer points in $\mathbb{Q}$-algebraic groups. There's a group $G_{\mathbb{Q}}$ into $H^{\circ}(X, \mathbb{Q})$. So I was told about Quillen's paper. I couldn't use it, the cardinalities are very large, so I wanted to find a small model, the rational homotopy type. You only need to go up to the dimension of the space. So you only look at a finite part, and then the matrices that commute with $d$ are a $\mathbb{Q}$-algebraic group. So you prove that this group is arithmetic and you give a more systematic statement about invariance, about diffeomorphism type. So you can start from a manifold, build an algebraic system and resolve your manifold up to finite ambiguity.

One tool is to develop another version of rational homotopy theory which can come from rational differential forms. I spent about seven years writing this. There were different points of view and I kept changing the viewpoint. My ideal picture of this course, topology, if you apply it in these other branches, it's quite interesting. It's more interesting in terms of how it arises outside. The main step in the paper is to say, you can't do the whole problem. If you don't soften it, you get stuck at every stage. When I was a graduate student, everyone said that the action was, they talked about spectra, tensoring with $\mathbb{Q}$ is just chain complexes. Whenever there's a spectrum around they can really beat the hell out of it. A spectrum appeared in some moduli space, and they figured out which one it was. If you tensor them with $\mathbb{Q}$ they are just chain complexes. If you do unstable homotopy theory where you have the ring structure, then that nonlinear structure leads to a rich theory.

