

Dennis Seminar

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So I'll keep talking about the Eilenberg MacLane spaces, $K(\pi, n)$. When π is finitely generated you only need to worry about $K(\mathbb{Z}, n)$ and $K(\mathbb{Z}_k, n)$. Recall $K(\mathbb{Z}, 2)$ was \mathbb{CP}^∞ was the infinite symmetric product of S^2 . So $K(\mathbb{Z}, k)$ is the infinite symmetric product of S^k . You can think about \mathbb{CP}^n as the coefficients of a polynomial, and then that gives you the roots sitting in $\bar{\mathbb{C}}$. So $SP^\infty(S^k)$ is $K(\mathbb{Z}, k)$. I used this to prove something a long time ago. There's this natural cohomology class, an element in $H^k(K(\mathbb{Z}, k), \mathbb{Z})$, here you have $H^k(_, \mathbb{Z})$ maps onto $Hom(H_k(_, \mathbb{Z}), \mathbb{Z})$ and in this case it's an isomorphism, where $\mathbb{Z} = \pi_k \rightarrow H_k \rightarrow \mathbb{Z}$. When $n = 2$ there's a cycle, you take $S^k \times S^k / \mathbb{Z}_2$ so you look at the set of antipodal pairs. It doesn't intersect the diagonal. When you take the quotient you get manifold with a singularity that looks like a cone on a projective space. So in the manifold part this has a dual cohomology class, which is this fundamental class. You can see this pretty clearly when $k = 1$, then this is the Mobius strip. You have the boundary as the "singularity" and the zero section as the antipodal set. There's the Thom class of the normal bundle, which counts crossing transversally.

In fact, let me go back and discuss the Thom class. Assume you have a vector bundle. This is a good application because it shows some obvious naturality properties if you understand obstruction theory. So look at the obstruction to having a cross section in the sphere bundle, if the sphere is a $k - 1$ sphere it is in $H^k(\text{base}, \pi_{k-1}(S^{k-1}))$. We can pull the vector bundle up over itself. The pullback bundle of this disk bundle has a canonical section, where you take the point which is that point in the disk bundle. So it's a nonvanishing section on the bundle over the sphere bundle. What is the obstruction to extending this canonical section to the entire disk bundle? It's in $H^k(\text{Disk/Sphere Bundles}, \pi_{k-1}S^{k-1})$.

This is a relative homology class, which when restricted to a fiber disk modulo its boundary is a generator.

The obstruction to extending the identity on the sphere over the disk is the generator of the relative homology of the disk.

You can apply this construction to a normal bundle which will live on a neighborhood and vanish on the boundary. This is a nice union of the first and second semesters, it's a nice

picture of a cycle and the idea of cohomology as obstructions.

If you have a submanifold you have the tangent space to the big space and the tangent space to the subspace. The normal bundle is their quotient. Using a metric you would make this orthogonal so that you can get this tubular neighborhood. So you exponentiate the normal directions and get a little tubular neighborhood. If it's closed but not compact you can still do it. I want this to be a cycle, a closed manifold. So this is a cool thing. You get two things from the submanifold. It itself is a cycle inside the space. You can also look at the normal bundle and its Thom class, and that gives you a relative cohomology cycle in the complementary dimension. There's the theorem $H_k \cong H^{d-k}$ for manifolds. So H_k is represented by a submanifold and H^{d-k} by the Thom class of the normal bundle. That's an extra feature that's not stated. The Thom class is in the relative cohomology. It vanishes on the boundary. It's like a function, it can be extended by zero. It's defined on every space that contains this thing. If the manifold has boundary the Poincaré thing says that the absolute homology is the relative cohomology.

I have this vague general conjecture that the diagonal in $M \times M$, the detailed structure of the Thom class of this thing has all of the details of the deep structure of the manifold. For example if you apply the Steenrod operations you get the Stiefel Whitney invariant of the manifold. It really uses the fact that you have a manifold.

So here is the set of antipodal pairs inside the symmetric product of the sphere. If the two points are antipodal, they are nowhere near the diagonal, the singularities. There is a Thom class away from the singularity. The diagonal class is not as good as this class A , it's twice this class. And A has a normal bundle, which has a tautologous class as being in the restriction of the fundamental class in the Eilenberg-MacLane space. I wanted to think about this in $S^k \times S^k \times S^k$. Anything with a normal bundle has a Thom class.

Oh, I've been ignoring the basepoint. [Some discussion]

Say $g, f : S^k \rightarrow S^k$ of degree one. Then there exist two points x, x' so that their images under f and g are antipodal. Let's use this Thom class idea. Suppose not. Look at the induced maps G, F on the symmetric square of S^k . Then in there we have the set of antipodal pairs. The two maps are homotopic because they are homotopic. Note that f and g are homotopic, so F and G are homotopic. Now we have this set A which has this Thom class U_A . Mark asked, is this a global class, yes because you extend it by zero to the whole space. These are unordered pairs, it doesn't intersect the diagonal, it's the antidiagonal. Maybe A works in general. Anyway, so $G^*(U_A) = F^*(U_A)$. Suppose $F^{-1}(A)$ is disjoint from $G^{-1}(A)$. So every pair of points made antipodal by f are not made so by g .

So suppose this is the picture of F and G mapping into A . So this pulls back and you get two classes in the pullback. So if they have disjoint support, the product is zero. Then $V \cup V$ is zero.

But I claim $U_A \cup U_A \neq 0$ in $H^{2k}(SP, \mathbb{Z}_2)$. Let me use this to finish the proof. Since the degree is odd, the degree on $SP^2 S^k$ is also odd. It's a cycle so you can talk about degree. The top homology is nonzero in \mathbb{Z}_2 . So both things should cup with themselves and be nonzero.

This is the perfect application of this course. When k is even you can just use cohomology directly.

Use the standard flow field on a sphere to take an antipodal pair, you destroy every antipodal point, except the source/sink pair. So there's one intersection. There's a deformation of the space A .

The day before yesterday this was someone's research paper. He had this complicated proof. He's a geometer. He has a wonderful geometric proof that's part of another discussion, he thought, this has to be a known fact. This is showing why it's known. It follows from manipulating the machinery of cohomology. You've heard of these Borsuk theorem. There are always two antipodal points with the same temperature and pressure. The two quantities define a map of the two-sphere into the plane. Otherwise there's some kind of funny thing going on, a map without a fixed point or something. I like this one you can prove to a grade school kid, this guy walks up to the cabin and walks down the next day, and he leaves at the same time. He has to be at some point on the path at the same time both days.

$[X, K(\pi, n)] = H^n(X, \pi)$. Fact: $K(\pi, n)$ are unique up to homotopy type. That can actually be proven from what we said. So let's consider spaces with two homotopy groups. We can add cells to get it down to one. Then we turn this into a fibration and the fiber will be the one with the other homotopy group. So there's a long exact sequence, but things map isomorphically down and the groups are in two different dimensions. So a space with two is naturally a fibration. What is the obstruction to cross section? This fibration is completely determined up to homotopy by the first obstruction to cross section. This is an element in $H^{n_2+1}(K(G_1, n_1), G_2)$. This is a fact. It's certainly an invariant of this.

You can do this with three nonzero homotopy groups. So then there's an invariant for the next one. You see an inductive picture. You pick the space O_1 , use the first obstruction and build up O_1, O_2 , and so on. If you have any space X we can kill all but its first homotopy groups by adding cells and getting O_1 , and then you can get a sequence of fibrations to $O_1 \dots O_n$. This is called the Postnikov system of a space. You have O_1 and then O_1, O_2 . This map is an isomorphism of homotopy groups as high as you considered, so you can study any finite dimensional problem about X .