Dennis Seminar
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All right, there we are, should we do some branched covers today? Nicholas, a former student, [unintelligible]branched covers. Maybe we'll go over those presentations next time, everyone has some little things to smooth over. So let's talk some about the building blocks of topology, of homotopy, the Eilenberg-MacLane spaces $K(A, n)$ where for $n \geq 2$ A is Abelian and for $n = 1$ it is anything.

The first fact is that there exist spaces $K(\pi, n)$ where $\pi_n(X) = \pi$ and $\pi_i(X) = e$ for $i \neq n$.

Examples? S^1 is $K(\mathbb{Z}, 1)$ and T^n is $K(\mathbb{Z}^n, 1)$. These are all of the finite dimensional examples. Not really, a counterexample is that you can map S^1 to itself by degree two and form the mapping telescope of this map, then the union of all these spaces is $K(\lim \mathbb{Z} \to \mathbb{Z} \to \cdots, 1) = K(\mathbb{Z}[1/2], 1)$. These are all the compact finite dimensional Abelian $K(\pi, n)$. You can get any subgroup of the rationals in this manner.

$\mathbb{RP}^\infty$ is $K(\mathbb{Z}/2, 1)$. This is doubly covered by a contractible space.

Non abelian $K(\pi, 1)$ are, e.g., "most" three-manifolds. But π is not Abelian. Any negatively curved, in fact, you need only nonpositively curved manifold is a $K(\pi, 1)$, because the universal cover is nonpositively curved so there's a unique geodesic between two points. So there are a lot of $K(\pi, 1)$ for π nonAbelian. For example, any Riemann surface is a $K(\pi, 1)$ except the two-sphere. It's a big conjecture, not proven that any two $K(\pi, 1)$ for the same π are homeomorphic. This is known in the hyperbolic case, and above dimension two they are isometric. They are known to be H-cobordant.

Any graph is a $K(\text{free group}, 1)$. Any presentation of a group, you have generators and relations, and you take the n generators and attach two-cells according to the relations. I don't know for which presentations the higher homotopy groups are zero. If π_2 is zero, then the universal cover is simply connected, it's two dimensional, and H_2 is zero, so it's contractible. So $\pi_2 = 0$ if and only if the space is a $K(\pi, 1)$. This has something to do with the relations. Anyway, the proposition is that there exist $K(\pi, n)$ spaces (unique in the homotopy category) for every π. I was about to tell you. There's one more nice example, $\mathbb{CP}^\infty$ is $K(\mathbb{Z}, 2)$. You get this from the long exact sequence because of $S^1 \to S^\infty \to \mathbb{CP}^\infty$.

These are all the nice examples I know, I was going to say, but there's this Dold-Thom theorem that gives a nice representation.

Oh, there are also the lens space things, you get infinite lens spaces which are $K(Z/n, 1)$. We can always take products to get direct sums of these groups. For $\mathbb{Z}/n$ in the second case and for both free and torsion parts in the higher dimensions we have question marks.

Any group with torsion has an infinite dimensional $K(\pi, 1)$. This is a fact, you can prove this. This is easy to prove. Suppose the contrary, so you have a finite dimensional space which is contractible and the group π acts on it freely. So $\widetilde{K(\pi, 1)}$ is finite dimensional, so π acts freely on a finite dimensional space. Then a torsion element α acts freely on a finite dimensional space so this finite dimensional space modulo $\mathbb{Z}/n$ is a model for $K(\mathbb{Z}/n, 1)$. So take $\mathbb{Z}/2$. We have another model with cohomology in infinite dimensions, so since these are unique in the homotopy category this finite dimensional model would have to be infinite dimensional.

On the other hand, let's go ahead and prove, get some picture of $K(\mathbb{Z}, n)$ and $K(\mathbb{Z}/k, n)$ in these higher dimensions. Let's get $K(\pi, 3)$. I know something about the homology. H_1 and H_2 are 0 and H_3 is $\mathbb{Z}$. Then H_4 is zero because it's surjective in the next dimension. Well, so we could start out with an approximation, S^3 is a space that looks like this. Now one thing you could do is just go to the first place where there's a nonzero homotopy group and add a cell to kill it. Then you need to add higher cells to kill the new higher homotopy groups. You keep attaching cells to kill the homotopy groups, and build $K(\mathbb{Z}, 3)$. This is fairly elementary to prove that this produces all the Eilenberg-MacLane spaces. But it's mysterious, we can't compute homotopy groups.

Now these cells that you're attaching, you could imagine how many there are in various dimensions are basically the chain complex for the homology. For each free part in the homology you need a cell, and for each torsion part you need two cells. There was a famous seminar where they computed the homology of these, this has torsion of all orders, it's infinite dimensional. Rationally you know it, though, it's consistent to ignore torsion, then they look like S^{2n+1} is $K(\mathbb{Z}, 2n + 1)$. All you're doing with the homotopy groups is killing torsion. It looks like the S^1 case. The higher odd spheres work up to torsion. The building process is complicated so life isn't over even if these are hard. We know one even case, $\mathbb{CP}^\infty$. For n even, we have a $K(\mathbb{Z}/2)$. We have one class in degree two and then all of its powers. It's $k[u]$. So for n even $K(\mathbb{Z}, 2k)$ looks like $\mathbb{CP}^\infty$ ignoring torsion.

To get a space you can take Eilenberg-MacLane spaces and multiply them, but that's not the only way. I asked a student, George Cook, when I was working on my thesis, it's exactly when the Hurewicz homomorphism is an injection onto a direct summand. So you should be able to identify some smart graduate students.

$H^n(X, A)$ is in one to one correspondence with $[X, K(n, A)]$. To build a map you need to build cohomology classes. So cohomology classes give you maps. Define a map using obstruction theory and then the other way by a direct obstruction. So suppose we have a map $X \to K(\mathbb{Z}, n)$ and the point map. Try to construct a homotopy. I want to extend it over the interval. I can go up to the $n - 1$ skeleton. Oh, we were always discussing cross sections of fibrations, but if you take E to be the fiber cross the base, then constructing a cross section is the same as studying a map into the fiber, the trivial bundle. Instead of getting an existence, I get an existence of a homotopy problem. I get $H^{i+1}(X \times I, X \times \delta I, \pi_i(F))$. This is the same as adding a cone on the boundary, so it's the same as the suspension, so it's $H^i(X, \pi_i(F))$. are the obstructions to homotopies between maps. This obstruction is the obstruction to homotoping to zero, and it's the map in one direction. Conversely, given this model of $K(\mathbb{Z}, n)$ I can map each cell over the n-sphere the appropriate number of times.

Now I want to go to the $n+1$ skeleton. The sum of the four faces of the tetrahedron is zero because it's a coboundary. What is the obstruction chain, why is it a cocycle, what does it mean to vary it by a coboundary? That's what you should learn this semester. Because it's a cocycle you can extend it up a dimension, and then the rest of the way is for free because $\pi_{n+m}(K(\pi, n)) = 0$. You are pulling back the canonical element in $H^n(K(A,n), A)$ which is the inverse of the Hurewicz homomorphism.

A corollary of this is, if, for those spaces where the Hurewicz homomorphism is a split injection, those spaces are products $\prod K(\pi, n)$.

All right, so, let's see. So now, since, I was about to say that these are the only nice examples, but once I needed to know something that involved $K(\mathbb{Z}/n, k)$. There's a nice theorem by Dold-Thom, there is a nice geometric model, to motivate it, you first observe, think of complex polynomials. I can pass to the roots and get n unordered points in $\mathbb{C}$. If I multiply the coefficients by a complex number, that doesn't change it. That gives a map to $\mathbb{CP}^n$. If the first coefficient goes to 0 then the root goes to ∞. Then this becomes a bijection. Then $\mathbb{CP}^n$ is the n-fold product of S^1 divided by the symmetric group. The fact that this is even a manifold is unobvious. For any three or one manifold this is not a manifold. Only in dimension two does the symmetric product preserve being a manifold. We could say $\mathbb{CP}^\infty$. We get the homotopy groups of this space, coincidentally the homology groups of the other space. If you take any space then SP^∞ has homotopy groups which are the homology groups of the original space X (isomorphic in a natural way). This is a nontrivial theorem with a lot of geometric content. For example, we can say $K(\mathbb{Z}, 3)$ is the infinite symmetric product of the three sphere. A point in $K(\mathbb{Z}, 3)$ is a bunch of points in $K(\mathbb{Z})$.

I was going to suggest why this was true using branched coverings. The idea is that a cycle in X is the same thing as a sphere in the high symmetric product of X. I want to say that cycles on one side are spheres on the other. There's a James Bond movie where a villain gets a message that something bad has happened, and he makes a chess move, bam, and then a minute or two later everyone realizes it's checkmate, a totally unsurprising move.

You take things and send them to $\mathbb{R}^n \cup \infty$. The linear extension, at the level of edges, might map to the outside, depending on the orientation. That gives a branched covering over the n-sphere. If you have a map into X, that's a multivalued map of the sphere into X. Cycles are really multivalued maps into spheres. Conversely, a map to points in the sphere, that gives a branched cover, there's your cycle. This is somehow, you can trade homology in X for homotopy in the symmetric product, and for homotopy, a branched covering over the homotopy is a homology.

I'm going over again, I'm not supposed to. Next time I'll say what you can prove with this thing. We are going to use only obstruction theory and a picture of a cycle.