Sullivan Notes
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I wonder if I should organize things a little differently.

I'm going to talk about the homotopy category, some of the basic properties, I want some of you to give presentations about the basic points, all right, who doesn't have shoes on, who isn't registered for the course, your participation will be voluntary. Yours will be involuntary.

This homotopy category that was kind of born fitfully in the history of topology has features that a lot of other categories don't have. In what Vincent works on, you don't have homotopies explicitly. The equivalences have some structure, people talk about these things, categories with higher equivalences. In differential geometry you don't have homotopy, but when you go to a more physical picture, these ideas come into play. It's good to know how it works because then these other categories can use these ideas, and they'll become more familiar.

It's also not exactly geometric and not exactly algebraic, it's somewhere in between. In my own education, since the first thing I learned about the homotopy category, I didn't learn a bunch of other things because they're described by the homotopy category. I can't give you definitions of triangulated categories, resolutions, those are already natural to the homotopy theorist. The algebraic structure is quite interesting. A famous mathematician, Daniel Quillen, who invented model categories, they're very useful for all kinds of algebraic problems.

Gabriel was just at a conference, there was a bunch of geometry and homotopy theorists. The language was different from what you're used to, right? It's close to arithmetic and algebra these days. The history, I mentioned a lot last semester, when they defined these with the cellular definition, they had a hard time showing they were invariants, then they showed they were homotopy invariants. Let $\mathscr{H}$ be the category of topological spaces and continuous maps where maps are taken up to homotopy. So $f, g : X \to Y$ are homotopic if there is a map $H : X \times I \to Y$ with $H(0) = f, H(1) = g$. You should specify whether or not the spaces and maps are based. Based means every space has a base point and every map sends the base point to the base point. There are really always two discussions, whether you have the base point or you don't, and it makes a big difference. Most of the time you assume spaces are connected and you have base points, but you can't always do that. The loop space of a non-simply connected space, for instance, is not connected.

[If they're based, do you require based homotopies?]

Yes, if it's based everything is based.

So homology will be a functor from this category. So if $X = S^1$ and Y is connected then $[X, Y]_{based} \cong \pi_1(Y)$. In the unbased case this is the conjugacy classes of $\pi_1(Y)$.w

There is an interesting theorem by a logician

Theorem 1 *(Freyel)*
Groups and homomorphisms correspond to the based case. The unbased part corresponds to groups and conjugacy classes of homomorphisms. This (unbased) category is not a concrete category, so you can't make it a category of sets and maps between sets.

Suppose $Y_1,\ Y_2$ have basepoints, you can always homotope your basepoint around. If you did this some other way you conjugate by the loop.

The fact that it's not concrete means you can't attach labels and so on, you can't line these things up, it's more complicated, it's twisted.

It's important whether to specify this or not.

So I've said what the objects and the arrows are. So I have a category. Let's talk about the based category. What is an isomorphism? It's a pair of maps where composition is the identity in the category.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

with $gf \sim I_x,\ fg \sim I_y$. Normally you know what isomorphisms are. When I talk to physicists, they don't define things. But you go and have a beer with them, ask them when two theories are isomorphic, you start to get a picture of a definition. There is some equivalence due to Thirring.

We're looking at CW-complexes, a subcategory $\mathscr{C} \subset H$. These are defined inductively. In dimension zero they are any discrete set. In dimension n you attach a discret set of n-cells to the $n-1$-skeleton. These are kind of like free objects in the homotopy category.

I'm going fast because some of you who are presenting some pieces, well.

The topology is that a set is closed if its intersection with every open cell is closed. This CW, it's kind of unfortunate, this stands for weak topology, but this is the opposite of how analysts use this word weak. This topology is very strong. A point in every cell is a discrete set.

You two are unregistered, right? You have to sit over there, I'm just kidding. We're going to have class participation, and the registered participation is involuntary. So the cells are really spread out. We're interested in spaces which are equivalent in $\mathscr{H}$ to CW complexes. This is closed under lots of operations, taking loop spaces, and so on.

There's a nice theorem of Whitehead,

Theorem 2 *In $\mathscr{C}_{pt}$ equivalence of connected spaces is recognized by (set) isomorphisms of π_n.*

This is not true in general in $\mathscr{H}$. For example, you can map the topologist's sine curve to a point and it will be an isomorphism on all homotopy groups.

I need someone to present the proof of this theorem. Gabriel will present it on February 13.

Given any space X there is a functorial associated space $SX \to X$ where $SX \in \mathscr{C}$ and S induces an isomorphism of π_i. You can always find such a thing, this is called the singular complex, this is a natural construction. Why don't you give a proof that this is an isomorphism. The vertices are all maps of a point into X. The one-cells are maps of an interval into X, and so on. The boundaries attach along the appropriate maps of things.

There is an obvious map into X and then any homotopy can be triangulated. If it were homotopy equivalent to a CW-complex, this huge space is a model for it. It's sort of simplicial. You may have done some folding in. It's not actually embedded, but in this construction they are. These were called semisimplicial complexes.

Now, uh, this is due to Eilenberg. It took some kind of powerful abstraction to form a complex with such big cardinality. This defines a chain complex, and then the homology is singular homology. If you subdivide this thing a little bit you get a simplicial complex.

I'm not sure about that, maybe there's some degeneracies. It's not quite a simplicial complex, but the attaching maps can all, it's very controlled. Certainly, this is a geometric object, you're identifying things in very simple ways. Okay.

[You said that the idea of a simplicial set grows out of this? Is that because that's a simplicial set?]

Yes, but that definition came later. It started with the boundary operator. Now you would say that these are the n-cells and you have face operators by deleting an index and degeneracy operators where you project down and then map into X by some map. These things are closed under, this is called a simplicial set. The next generation doesn't know about the geometric roots. Then there is the geometric realization of a (semi)simplicial set. This is the thing I was just talking about, this space.

Now let's go to a notion of homotopy fibration. So Serre, around 1950 in his thesis, made a big advance. There were all these bundles and some fiber moving smoothly over it. These things were around in the 1930s and 1940s, and in his thesis he came up with a more general notion, a fibration. It had all the good properties of fiber bundles but now you could take all kinds of infinite dimensional spaces, loop spaces and so on, and he related the homotopy groups of the fiber, base, and total space in a spectral sequence.

I think this is true, for no simply connected non-contractible finite complex do we know all the homotopy groups.

Suppose you have, I'm only going to define this over, define

Definition 1 *$E \xrightarrow{\pi} B$ (connected base) is a homotopy fibration with fiber F if B is a regular cell complex (the boundary of each cell is embedded) and $\pi^{-1}(\sigma) \hookrightarrow \pi^{-1}\eta$ is an equivalence for each inclusion of cells $\sigma \subset \eta$ and (maybe) that the "fiber" F belongs to $\mathscr{C}$.*

There is a fact, Nathaniel suggested a path lifting property. When Serre defined this thing, it was Hurewicz actually, these things can be shown to have this property, but we'll get to that when it comes up. When you're making constructions, you can often see this. It's really all you need for anything.

[Will you say what a cofibration is?]

No, that was last semester. You can take $A \to X \to X/A$. That satisfies an exact sequence of homology, that's exactness. These satisfy one for homotopy. I may need the lifting property.

Here's a fact. Any map $\mathscr{E} \to \mathscr{B}$ in $\mathscr{C}$ is equivalent to a homotopy fibration.

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "\cong"] & \mathscr{B} \arrow[d, "\cong"'] \\ E \arrow[r, "\text{homotopy fibration}"'] & B \end{tikzcd}$$

So for every map there is a fiber to compute. These tough guys like Ib Madsen, they can tell you what the fibers are.

That's the theorem you kept hearing, the Madsen Weiss theorem, about the stable homology of moduli spaces. The last talk in the conference used that to analyze what a string theory was when the propagator was [unintelligible].

So what's the proof? Let's see if I can prove this without assigning more work. Let me call the spaces X and Y. Keep everything connected so I don't go crazy. Step one is to replace $\mathscr{B}$ by a regular cell complex. I'll take the singular complex, mapbe subdivided a little bit. Now replace $\mathscr{E}$ by the set of pairs (e, γ) where $e \in \mathscr{E}$ and γ is a path in B with $\gamma(0) = \pi'(e)$. Now first of all these are homotopy equivalent by retraction to the basepoint.

Nathaniel, why don't you prove this. We have to prove that E has the type of a CW complex.

We have a map $E \to B$. which is $(e, \gamma) \to \gamma(1)$. I claim that this is a homotopy fibration. So π^{-1} of an arc are the paths ending on the arc. You can squeeze down the arc. Take the paths to the point, and project the simplex back to that point. I have an inclusion $\pi^{-1}(\sigma) \hookrightarrow \pi^{-1}(\eta)$. Now I need to deform the bigger space back to the smaller space. Add a little path back to the endpoint. You have to organize some details but it's intuitively obvious. We need someone to cover this in more detail. Nathaniel, we'll say February 13 and see how it goes. Converting maps to fibrations, they're not geometric any more, they're usually infinite dimensional.

A consequence of this definition of homotopy is an action of $\pi_1(B)$ on the "fibre." I keep this in quotes because all of these are homotopy equivalent. You can choose a path between any two points and you get a well-defined homotopy equivalence between two paths. If you move by a homotopy you get the same equivalence. If you move the loop by the homotopy, you get the same automorphism.

Suppose I wanted to have the fiber be based. I want a point above each of these so that the inclusions, I'd need some sort of homotopy between the base points. There's the notion of a section of a fibration. Next time we want to investigate whether we can build a section, and that will lead us to cohomology.