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Gabriel C. Drummond-Cole

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So, over the one-skeleton to get a cross section we associated a covering space, to each piece we associated the component of that piece, and you needed a trivial cover, which would extend to a cover everywhere. So then we had a connected fiber and a section over the one-skeleton, a complicated thing looking like this sitting up in the total space. In the total space we want to extend over the two-skeleton. You want all the loops you've brought up to be nullhomotopic. So you can change edges but then that will affect what happens to each one of the loops incident on that edge. So this gives an obstruction in something like $H^{2}\left(B, \pi_{1}(F)\right)$.

You have something assigned to each face in the ordinary cohomology. Then you say the sum of the boundaries are equal to zero, that's the cocycle condition, and then there are trivial cycles. Every edge appears twice with the opposite sign, so things cancel and you get zero. That uses the abelian condition.

Now I thought over the weekend that maybe what I should have done was suppose I had it up over the two-skeleton and extend something, where the obstruction will be $H^{3}\left(B, \pi_{2}(F)\right)$. So one way to do it would be to do the Abelian case and then come back down. Or we could just struggle through the storm and be happy when we get to Florida.

Things we did last semester are more geometrical, this semester are more algebraic. Last year we had the idea of geometric cycles representing homology. Did we talk about branched coverings last semester? Think of $p(x, y)=0$. You get a polynomial in $y$ with coefficients polynomials in $x$. As you vary $x$ you get, generically, distinct roots, but there are degenerations where the roots coincide. So then fibers are in cyclic groups, and you cone off around the points.

The locus it's branched along is codimension two. Every oriented 3-manifold is branched along a link in $S^{3}$. You didn't prove this, but every oriented 3-manifold is a three sheeted branched cover along $S^{3}$. This is conjectured, that a $n$-manifold is an $n$-sheeted cover along $S^{n}$, which is known through four.

So you could get a graph. Now the curves can come together and you get a two-sphere over
here. You have to cone that off. There are parts around each line, and then the informations come together at the vertices and fit together to give you something. The branching locus of a branched cover is what kind of object in topology?

Take $\mathbb{C P}^{2}$, then the quotient by conjugation is $S^{4}$. The branching locus is $\mathbb{R P}^{2}$. One answer is that, I'd like to study branched covers, what if you want to deform them? Is it something like homology or homotopy? It should be homological in nature, the answer. If your branched covering, if you move it over an interval the locus can move around. You want to assign group elements to each piece of the locus. This will be like talking about $H^{2}$ by Poincaré duality. Certain kinds of branched covers are what is called cyclic. These can be described by cohomology. But I want a description of the general one. Note that in the example, the locus is nonorientable but both the total and base spaces are.

I noticed in 1990, I think, that I didn't understand what a branched cover is, nor did anyone else. One of my graduate students wrote a thesis on branched covers. Somehow it's a very appealing thing. It was the start of abstract geometry and topology. A holomorphic map between manifolds of the same dimension, it's reasonable to want to get a branched cover.

Alexander proved that every manifold is a branched cover. You replace a point in the sphere with a multivalued point. This gives a nice picture of every manifold. As I say, for three and four manifolds, you can take the number of sheets to be three and four.

If you don't like that I'll offer you something else. Take Riemannian geometry. What's the principal character in Riemannian geometry? What's the principle character in Riemannian geometry? Have any of you had an elementary course in Riemannian geometry? This is in Spivak. Riemann was a student of Gauss. Gauss invented the Gaussian curvature. He used it to show that you couldn't make maps of arbitrary surfaces without distortion, up to scale, and that you can make maps that distort the scale isotropically in each direction. Riemann had the sectional curvature, had a global version of both of these. You have $\binom{n}{2}$ choices, these fit in a matrix. This is more than a number. For each two-plane you have a skew-symmetric matrix that rotates by the amount given in the curvature.

I noticed that I didn't understand the Riemann curvature tensor. Well, there are like too many parameters. One way to look at it is, it's a two-form with values in the Lie algebra of endomorphisms of the fiber, the tangent space. There's a curvature tensor for any bundle so we say it like that.

This is another thing I didn't understand. Differential geometers make hypotheses, then they do a bunch of computation and find out that something is zero or something has a sign, and that's a theorem.

So for example, give a free set of moduli such that every three-manifold looks locally like one or another of them. The diffeomorphisms act on the metric, you are interested in the quotient.

There's a physics, degrees of freedom argument. Let's go back to what Arnold alias Andrew said. You get a positive definite symmetric matrix with $n+n(n-1) / 2=n(n+1) / 2$. You have this many functions. Now, a diffeomorphism does what? There are $n$ functions of $n$
variables. So you want to take this number and subtract $n$ and you get $n(n-1) / 2$. This is how many entries you have in the Riemann curvature tensor. It's not true that if you know the curvature tensor, you know the metric, well, the statement is subtle.

In the 1990s I noticed that these two problems are sort of equivalent. In fact, what you can do, I was thinking with a general curvature tensor, take a very fine triangulation. Not only would I like to understand this, I'd like a combinatorial version of this Riemannian geometry. In 1970 I wanted to do this because, considering the tensors $\Omega^{\otimes n}$, taking the trace gives you ordinary forms, these are characteristic classes. The differential geometers have formulas for them and I don't, that makes me jealous. A lot of people like this problem, you could get a job at Renaissance technologies. Anyway, this area was kind of very interesting, and then in the past twenty years with string and quantum theory, the desire is to make one quantum theory which has in its limit both the Einstein relativity, which is gravity, and, well, the quantum field theory describing the other three forces. The big intellectual problem, intellectually, is to have one theory to describe all these forces. A combinatorial version of Riemannian geometry are difficult. If you could somehow chop off the infinities of fine scale, that would be easier. If you had that machinery going for you it would help. That's the current motivation for wanting to understand Riemannian geometry from a combinatorial point of view.

Wilson took part of this and make a combinatorial guess, and for the parts he was interested in, it worked, he got fairly good agreement with the phenomenology, and that's lattice field theory. They don't know how to do spinors very well. There are fragments of truth mixed with leeches and so on. If you took a very fine triangulation and you took your curvature tensor, this a matrix of two forms and this, a connection, is a matrix of one-forms.

You may think this is a waste of your time, but let me tell you, mathematicians and physicists, neither understand this. A connection is a non-abelian one dimensional thing, and the connection is like the coboundary, it's not linear any more. Then there are fancy formulas. If you did this with complex bundles you'd get the Chern classes. $\operatorname{tr} \Omega^{*} \Omega$ figures in the Yang-Mills action.

So you get some kind of thing around the branching locus, this is last semester. The curvature is a cohomological something like this, I wanted to connect the two. The idea of a flat connection, if the curvature is zero, if there's a connection, which gives you a horizontal subspace, which lets you lift paths, and when you go around a loop in the base, you get a shift in the fiber. If you take this smaller and smaller, to the derivative you get an infinitessimal shift. If the curvature is identically zero you come back to the same thing. You get sheets lying over the base, you get a foliation where each leaf is a covering space of the base. If you look at a branched cover, that's what you have. Outside the branched set you have a discrete set, and it's like the curvature is concentrated on the branching locus.

So you can always get this foliation over the one-skeleton, and then you can fatten this up until you get the complement of the codimension two part. It will look flat on this open set the complement of the $n-2$-skeleton. Going around a piece of this is no longer a permutation of a discrete set, now you have an uncountable fiber.

You could have the orthogonal group, the sphere, and a rotation of the sphere.
So these two geometric things lead you to something that wants to be $H_{n-2}(B, *)$. That's where I got stopped a few years ago. I didn't know what the non-Abelian coefficients meant.

We take this natural problem, and bingo, at the two-skeleton, we get an object like this. We can look at the structure and try to understand it, and in the process we will be understanding something about these other things.

There is some kind of non-Abelian, in Riemannian geometry, they would never write that the curvature was $H^{2}(-, *)$ or in the branched cover that something is $H_{n-2}(B, *)$.

These are dual, they're, another aspect is that in a manifold you can't really distinguish the two. What I'm claiming is that we can go back and study the obstruction to getting an obstruction over the two-skeleton and get some idea of what this is. Usually in the books they skip this and express this in terms of fundamental groups of the space and the fiber. A special case of what we're trying to do, I've got this homotopy fibration and the part over each face of the cell including into the cell, that's a deformation retract. A very easy definition, And, uh, we've gotten down to the case where all the fibers are connected. So if the fiber is connected, I claim tha the map $\pi_{1} E \rightarrow \pi_{1}(B)$ is onto. Because I have a cross section over the one-skeleton. I can take a loop and pull it up, so the map is onto. If we could build a cross section over the two-skeleton, that would give us a map from $\pi_{1}$ of the two skeleton, I'd get a map back because above two there is no new information in $\pi_{1}$. In the usual texts they suppose you have the map back and go from there. If we're not assuming this, we're, given any groups and a surjection, you can build a space that realizes that, we're studying a hard problem of finding a lifting of a surjection of non-Abelian groups. I've seen fragments of this theory deep in group theory. If the kernel is Abelian, there's an obstruction in $H^{2}\left(\pi_{1}(B), A\right)$. Conjugating by the part in $A$ doesn't do anything, and you have a module over $\pi_{1}(B)$. They might have a discussion in the non-Abelian case if they specialize in group theory from end of the nineteenth century.

You look at $\phi(x y)(\phi(x) \phi(y))^{-1} \in A$ which is an obstruction for a set section $\phi$ to be a group section.

Somehow if this abstract thing is unsuccessful, it means the Riemannian geometry has found its way through with calculus, but I hope you can understand this. So we have this three-cell in the base, and we have this, flatter, down here, and we have a lift on the one-skeleton. Then these are filled in with two-cells and we want to know if we can do this above. So we get an element in the fundamental group of the fiber based at various points. We can take an element and move it over. We can move it around. Now, what equation is satisfied by all these elements? The equation will be that every time we have cells that bound a three cell, there will be an equation. Suppose we have an actual tetrahedron, then we have four "fields of elements" (not four elements because they exist independently at each basepoint). So we have some loops in the space, we have four loops in the space and the product going around the first three is homotopic to going around the back face. That's the equation.

Now maybe you can think about it, we want $H^{2}(B, *)$ and this will be cocycles (fields
satisfying this equation) modulo trivial cocycles. This is not a group. This will be a set with the trivial ones acting, and the quotient will be the orbits.

You can think of this as well as I can.

