Sullivan Notes
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So I have a regular cell complex B and then I have a space E mapping to B and this is a homotopy fibration, which means that over every cell, the preimage, over every cell, the part over the face included into the part over the cell is a homotopy equivalence.

Now we want to try to build, well, the problem is to build a section. I want $\pi \circ s(\sigma) \subset \sigma$. So over the vertices, are there obstructions? What if one of the fibers is empty? If one fiber is nonempty then they all are. So that's kind of minor. Over each component, well, this hypothesis implies that it must be surjective on the vertices, so we can do it over the vertices if B is connected and E is nonempty.

No one said that the fiber is connected, so it could be that we have chosen points over each fiber which cannot be connected with a path. So we could add that the fiber is connected, but let's analyze this a little more. Consider applying π_0 to the fiber. This is the set of path components. Replace over a vertex the set of path components and so on. Over every point in the edge put π_0 of the fiber. Over a vertex you get a discrete set. So you get a covering space. Over every point of the edge you get a block of discrete points, so you get a stack of these. As you go around, triangles have to close because of the equivalence to the fiber over the solid triangles. If you have a covering space defined over the two-skeleton it extends uniquely over the rest of the space. This is because the boundaries of the higher spheres are simply connected.

So we have a natural covering space here. Now any covering space may be thought of, connected or disconnected, with discrete fiber D is equivalent to giving an action of $\pi_1(B)$ on D and then forming $\tilde{B} \times_{\pi_1} D$.

So when does this have a cross section, in terms of the action of π_1 on D.

If you have a cross section, then going around a loop gives the same point. We have a section if and only if $\pi_1(B)$ fixes some point $s_0 \in D$. There is no action on the component of the covering space because this is not regular. If the total space of the cover is connected then this is true.

An example would be the wedge of two circles. The triple cover is connected so the action on D is transitive.

We've analyzed over the edges. The edges doesn't give you π_1 of the base. This gives you the generators but not the relations. If there is a point fixed over the one-skeleton, then there will be a section of the component over the two-skeleton. If we have a covering space and a section over the one-skeleton, then you can lift the triangle to one sheet, and then you can just fill it in. This is because the fundamental group of the one-skeleton maps to the fundamental group of the entire space.

So if we get our section over the one-skeleton, that picks out a component, and we can pick out a component over the whole space, so we can get down to a connected fiber. A section of π_0 of the "fiber" over the one-skeleton implies a unique section of the π_0 fiber over all of B. So now we can restrict to the case of a connected fiber. That means that E is connected. There are obstructions at each stage, which we have analyzed. This was a question of a fixed point of a group action and this was, B was connected and E is nonempty. We could have just started here, with fiber, base, and total space connected, and we get a section over the one-skeleton.

We see a lot of structure here already.

The next step is new for me. Usually when they do this, we get some analysis, we wouldn't have needed that analysis if we assumed everything was connected. Usually they make another kind of assumption, like that, if you have a section, you have a homomorphism of $\pi_1(B) \to \pi_1(E)$. That takes care of the two-skeleton. I'm going to analyze this. I actually clarified this during one of the talks in Morelia. If I skip one talk, I have so much fun, I don't go back, that only happened once in 1973. Usually people skip over the two-skeleton by the fundamental group discussion, then it will become the twisted coefficients obstruction theory.

So we want to do it over the triangles. Let me erase all of this. Over the triangles this will be non-abelian H^2. The reason I bother you with this, I think this is related to gerbes and 2-categories, although nobody understands it. It's sort of defined by this problem. We're going to go around the triangle. Choose our sections over these vertices, and we get a picture like this. Now, obviously, what we want to know is whether the loop over the triangle is nulhomotopic. If we actually want to say, suppose we take a different section over an edge. Then I'd change the section up to homotopy by a loop, π_1 of the block at either edge. That will change the obstruction of every triangle coming into that edge. So can you get a set of zeros everywhere?

That leads you to cohomology. This is the geometric source, a little tricky because the coefficients are in this non-Abelian group, π_1.

The guy made a reference to an impossible-to-read book in French by Giraud. So let's see what happens here. So if I want to calculate this in a fundamental group I need basepoints. By the way, the arc can also be used in a basepoint, any simply connected set, since two paths between actual basepoints are homotopic.

So we have an element in each fundamental group. I can carry this element of the fundamental group around, and I get xxx^{-1}, it conjugates it by itself. So to every triangle we have unambiguously defined an element of π_1 of the fiber. No matter which basepoint you use, they're coherent.

I want to change the homotopy equivalence statement to a deformation retraction statement.

If I take a function from the edges to the fundamental group of the fiber, then I can get a new path by going around these elements at the middle of each of the edges. So you can move these all back to beginning at the same point and you get a product of elements of the same fundamental group. You get an order too from tho order of going around your triangle.

This is our freedom, basically. We want to know whether we can make everything zero by using this freedom. This will lead us to cohomology.

Let's look at a tetrahedron. The claim is that this field of obstructions satisfies a cocycle condition, namely for each 3-cell (which I'll pretend is a tetrahedron), we have elements associated to three triangles touching a basepoint, and then a fourth one for the opposite face. Then the product of the first three, pulled back to the basepoint, is the fourth. That's the cocycle condition.

This argument shows the following. Suppose I just assigned a group element to every edge. Since over the tetrahedron I have this one block, I just assign to every edge a group element. Then if I form these products, I still get this relation. Next time I want to prove that this obstruction cocycle satisfies this equation.

This won't be a group, I haven't thought about its structure yet. The three-skeleton will be a lot easier, since π_2 becomes Abelian.