Sullivan Notes
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I was reading one of these newspapers, an advice column, it had a bunch of description of oral sex, it kind of shocked me, the generation gap is kind of large. Anyhow, I thought of this because I'm painting my teenage son's bedroom, there was a lot of stuff there I couldn't believe, but it's normal these days.

Let me start writing things. We have fibration, which can be described in different ways, geometrically (locally trivial) which leads to the path lifting property by choosing a point in the convex space of connections, which goes to homotopy fibration by the exercise, which goes to the categorical by obstruction theory (Mark, Andrew, and Ari).

[Can you go over a path lifting fibration again? I'm confused.]

Sure. Nathaniel is doing the composition from the path lifting property to the categorical. Somnath is doing the exercise taking the path lifting fibration to the homotopy fibration.

The path lifting one is due to Serre, you have a map from $K \times I$ to B, a homotopy, and you have a lift of $K \to E$ at time zero, then the homotopy lifts.

Hurewicz assumed you had such a structure, he added the property, suppose you have two homotopies, then the lifted homotopies should have the composition lifting to the composition of the lift. If you chop this up smaller and smaller, if there was differentiability, you'd have some kind of differential equation, and that's what the first arrow is.

I would say understanding this diagram and all of the proofs is half a semester, then you'd understand more about a fibration than some of the professors here.

All right, so usually when you're working, and you're not in topology, you'll get geometric ones. In topology you get homotopy fibrations. The categorical definition can be simplified and you have $(E, F) \sim B$ from the point of view of homotopy groups. Then we'd have this once Andrew gets his diagrams to commute and Gabriel shows us Whitehead's Lemma.

First there's a very general procedure called associated bundles (to geometric bundles). Usually homotopy bundles.

Oh, as you're digesting the diagram here you might want to prove to yourself that these are all strict inclusions. Only the last definition is categorical on the homotopy category. This difference between the topological and homotopy categories is interesting to me. This general procedure is, any time you have a geometric bundle, a map with a covering of the base so that over a neighborhood it looks like a product, that's a locally trivial bundle.

Oh, the geometric thing is only just barely for manifolds, if you're really a honcho you do this in the topological category.

Now the associated bundle is very interesting. Suppose you have a geometric bundle. You want to put more structure on the fibers. Then take all choices of that structure on each fiber, that's a new fiber. You have to be a little careful. If you took a tangent bundle you'd need to use diffeomorphisms instead of homeomorphisms and so on. When you have a family of complex manifolds it's not usually locally trivial because the complex structure changes continuously as you move along, but it is locally trivial as a differentiable manifold.

The first example is the tangent bundle of a manifold M. An associated bundle might be, well, examples of associated bundles are

a. The bundle of positive definite inner products on the fibers of T_M. The set of positive definite inner products on a vector space is a convex set so it's contractible. If you look at a triangulation of the base and want a section, look for obstructions there aren't any. So every manifold, as a corollary, has a Riemannian metric, well, if it's triangulable, there are some that are not paracompact that don't have Riemannian metrics. So also, anything cohomological you come up with out of this is a diffeomorphism invariant.

b. You have the bundle of orientations so this is a $\mathbb{Z}_2$ cover, and you have a section if and only if this has a section. You have $\pi_1 M \to \mathbb{Z}_2 = w_1 \in H^1(M, \mathbb{Z}_2)$ the first Stiefel Whitney class, which is zero if and only if there is a continuous basis of T_M on 1-skeleton of M. This is also $H^1(M^n, \pi_0(O_n))$. The ambiguity in identifying the fiber is a component of the orthogonal group.

   If you have a continuous basis you have an orientation of the one-skeleton. Then the cross-section lifts over the two-skeleton. If you have a cross-section of a covering space over the one-skeleton then you can extend it everywhere. If you have an orientation then that chooses a component, and then you can choose a cross section over both the zero section, and, because the group is connected, you can extend this over the 1-skeleton.

c. Almost symplectic, or almost complex. This means that on every tangent space you put a symplectic structure, a nondegenerate skew symmetric form. This forces n to be even. So if you want this nondegenerate you need n to be even. The fiber, the set of all symplectic structures on $T_M(x)$ is $Gl(2k, \mathbb{R})/Sp(k, \mathbb{R})$. Then for an almost complex structure, you put on every tangent space a J, an endomorphism satisfying $J \circ J = -id$. That would tell you, a complex structure is an action of the complex numbers. Then J acts as i, since you already have the action of the reals.

   If you're fancy like Blaire or Deligne you think about an actual complex structure but this is good enough for a country boy who doesn't need to do algebra, just using this picture. This is $Gl(n,\mathbb{R})/Gl(n,\mathbb{C})$.

These are almost complex structures because they're on the tangent space, not on the coordinates. If you can come up with this so it looks in the standard way with respect to a coordinate basis. This is equivalent to an integrability condition, if you, an almost structure is a cross-section, in fact, people usually don't discuss almost symplectic structures, it's a mistake, an almost, it's just begun, Gromov in '85 started talking about almost complex structures, you might think that Riemannian is an almost-Euclidean structure. There the integrability is that the curvature tensor vanishes.

In this case there is an obstruction, you can go to $O(2k,\mathbb{R})/U(k)$ so it turns out that the almost complex and almost symplectic structures are equivalent. You have to choose a background Riemannian metric, and then this little lemma, you can prove this, I'm not sure what the best way to say it is. Here the cross section is called ω or J and the integrability condition is, there are going to be obstructions, the $O(2k)/U(k)$ is even, only has things in even dimension, it's fun, we went over this in a previous workshop. On ω the condition is $d\omega = 0$. For J it's, $dJdJ + JdJd = 0$. In fancy words it's $d\bar{e}lta^2 = 0$.

For the Riemannian case it's that the Levi-Civita connection (the unique connection that preserves the metric) is flat. In terms of the metric, usually you denote the metric g, the cross section, and it's that some second derivative expression is equal to zero. You have something that takes two derivatives. If this is zero you have two coordinates where the metric is flat, you get a Euclidean structure, in the other cases you have completely analogous things. It's curious that it's only discussed in the case that you have $d\omega = 0$. The dual way to look at it so that it's quadratic. There's something called a Poisson structure. So you have $p \in \Gamma(\wedge^2 T)$, a section. So $\omega \in \Gamma(\wedge^2 T^*)$. So it's a skew symmetric form, but now we don't ask that it be non-degenerate. It can have varying ranks and these give a foliation, called the symplectic foliation.

The things in $\Gamma(\wedge^i T^*)$, the differential forms, they're a differential graded algebra. In $\Gamma(\wedge^i T)$ you have the Lie bracket as well, and you can extend this uniquely over the products so that the product rule is satisfied. This makes the whole thing into an algebra with a compatible bracket. So $[ab, c] = a[b, c] \pm [a, c]b$. This has the structure of a graded odd Poisson algebra. Somehow this bracket plays the role of d on this side. Another name for this structure is a Gerstenhaber algebra.

These are really fundamental structures, it's been around for a long time now. Somehow the older work wasn't emphasized. But with the influence of theoretical physics these structures are coming to the fore. Now each vector, when you have a non-degenerate inner product, gives a linear functional. So we can slam all these structures together, take them from one side to the other, get BV algebras. So what you can do, you can take the isomorphism between, define p to be $\omega{*}(\omega)$. So ω gives an isomorphism of forms to vector fields which extends to 2-forms to 2-vector fields, and now the integrability condition gives you $[p, p] = 0$.

If you have a Poisson structure then you can get the Poisson bracket on functions $\{f, g\} = \langle p, df \wedge dg\rangle$. There's an area of mathematics called Poisson geometry where this is the fundamental structure.

This isn't what I meant to talk about. Now there's some more, maybe before I leave this I should say something. This is a symmetric inner product, a skew symmetric, and an endomorphism. First of all there's this statement where any discussion without a quadratic form is uninteresting. Another statement, due to Rene Thom, is that the only important forms are one forms and two forms, well, include zero forms. So one-forms are like vector fields, two forms are like these things. The higher ones, generically, it's hard to classify them. You get moduli. Nevertheless, let's go on. I've been thinking a lot about quantum field theory and string theory. One of the most elusive parts is this Hilbert space that's supposed to be around. Another set of associated bundles, let's take an $\mathbb{R}^N$ bundle. Look at the associated Stieffel bundles, the k-tuples of linearly independent vectors. Look at the first obstruction to cross section. This is $H^{\ell+1}(\text{base}, \pi_\ell)$ where this is the first nonzero homotopy group of the Stieffel fiber.

When k is one, it's a nonzero vector. It's the vector space minus the origin, so has the homotopy type of the sphere. So ℓ is $N-1$. For $\mathbb{C}^n$ it's $2N-1$. So the first nonzero group is a $\mathbb{Z}$. In the $2N-1$ case the sphere is naturally oriented, so the thing is untwisted. The coefficients are $\mathbb{Z}$ and you get ordinary homology. When you come back around in the real case you might get a twist. Then in the case of $\mathbb{R}^n$, if it's oriented we get an untwisted $\mathbb{Z}$. In this case the first obstruction is called the Euler class in $H^N(\text{base}, \mathbb{Z})$. In the other it's the Chern class in $H^{2N}(\text{base}, \mathbb{Z})$, which is just the Euler class of the underlying real manifold.

Now, these are invariants because, ah, if you have two copies of the base and two partial sections, you can look at it cross I and you can extend the partial sections across $M \times I$. The obstruction to extending the section. Before you get to the first nonzero group, you can fill this in, you can do this across the middle too. If you started somewhere and you had it you could keep going. You can do the zero skeleton everywhere and extend until you get an obstruction. The first obstruction of the whole thing is on $M \times I$. That's why the first obstruction has meaning, because every obstruction can be fit into $M \times I$ and extended across. These objects are invariants of the bundle, not your choice. So then you can do, ah, now, what I got was that the Stieffel bundle was, let's drop the complex case and keep this $\mathbb{R}^n$ and do the $k = 2$ fields. If you put one vector in, it's a sphere. If I want to put a second vector in I might as well suppose it's orthogonal. This is $n-2$-connected, and so on. Let's say a space is k-spherical if it's k-connected. So the kth Stieffel bundle is $N-k$-spherical. The first obstruction is in H^N, the next in H^{N-1} and so on. These groups, we have to worry about what they are, they will turn out to be either $\mathbb{Z}$ or $\mathbb{Z}_2$ and so when we reduce we let these be $\omega_N \in H^N(*, \mathbb{Z}/2)$. If you did the same thing with the complexes you get a $\mathbb{Z}$ every times so you can do it with $\mathbb{Z}$, these are the Chern classes, but they only occur in the even dimensions.

To finish this discussion we need to show that these groups are really $\mathbb{Z}$. Suppose that $k = d$ is fixed. Then the k-frames in $\mathbb{R}^N$ is $N-k$-spherical. Then over some base B of dimension D fixed. So we have an $\mathbb{R}^d$ bundle over a base of dimension D. These two numbers are fixed. So let $N - d > D$. Then over the base of dimension D you put the Stieffel manifold of dimension N. So a cross section is going to be, sorry, what I put here is a torsor of this, the embeddings of the fiber, which is an $\mathbb{R}^d$ bundle, into $\mathbb{R}^N$.

What is the space of embeddings of $\mathbb{R}^d$ into $\mathbb{R}^n$? It's the Stieffel fiber. This tells you that for

$N > d + D$ there is a unique embedding of the twisted bundle into the trivial bundle. You can take the Grassmanian of d-planes in $\mathbb{R}^n$. Then you have the tautologous d-bundle. So you get a map across, and the Whitney theorem. It's much better to know this proof than the statement of the theorem.

I like this proof, you can adapt it even to, there's a well known statement in algebraic geometry, this statement isn't true, but the diagram works, you can use it to prove things in front of people who don't think this map exists.

We have our characteristic classes coming out of the existence of obstruction theory. When are we going to have this exposition, next time? And the hard one the following week?