Dennis Seminar
April 3, 2006

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February 18, 2011

Rational Homotopy Theory of Simple spaces (π_1 is Abelian and acts trivially on higher homotopy groups), this is an application of the Posnikov system.

Then we have this Posnikov system X maps into $X_{n+1} \to X_n \to \cdots \to X_1 \to pt$ where the maps are equivalences up to and including the indexed group. The fiber at level n is $K(\pi, n)$ where $\pi = \pi_n X$. Assume these are finitely generated.

So we have this idea that we can, this is kind of an algebraic picture of X. The first space is $K(\pi, 1)$ and then the next thing is a principal fibration determined by a three dimensional cohomology class which is the first obstruction to cross section. Then there's a spectral sequence that gives the cohomology of the total space, and then, except for not knowing the differentials of the spectral sequence, that's an algebraic picture.

When you have a fibration, like homotopy fibration, take the cohomology of the base with coefficients in cohomology of the fiber, it's like $H^*(B \times F)$ and in field coefficients this is something like $H^*(B) \otimes H^*(F)$. Then there's this video game called the spectral sequence. The E_2 term involves a bunch of things, there's a whole lot of shooting going on, and then homology of that is E_3 andid there's more shooting with longer guns, and, uh, this structure exists but it depends on the geometry what these are. For example if E is a point the video game is very violent and no one survives. If it's a product there's nothing going on, there's no differential. You take cohomology in the fiber direction and then in the base direction, and then you start trying to stretch classes through the fibration. This is an interesting structure, and it's an invariant of a homotopy fibration. The E_2 terms and then the differentials will be homotopy invariants. We'll try not to use it, really. That's the algebraic picture of homotopy type. So the first corollary, we can take this whole picture and tensor it with $\mathbb{Q}$. Then X_1^Q is $K(A_Q, 1)$ where A_Q is $A \otimes \mathbb{Q} = \oplus^i \mathbb{Q}$. The cohomology of this, you can't do this formally, you tensor your understanding with $\mathbb{Q}$. Now the cohomology of $K(A, 1)$ is $K(\mathbb{Z}^n, 1) \times K(\text{Torsion}, 1)$. The cohomology of the torsion part is equal to the cohomology of a point. There are lots of ways to prove this without explicitly knowing $K(\mathbb{Z}_n, 1)$. Then the free part is a torus and you get $\wedge(x_1, \dots, x_n)$. So $H^*(K(A_Q, 1)) = \wedge(x_1, \dots, x_n)$. To prove this, take a bunch of maps of the circles mapping by degree n into the next circles. The direct limit has $\mathbb{Q}$ in dimension one, and then you can just take the cartesian product of that with your space.

The next fiber is $K(B,2)$ and we put as the new fiber $O_Q = K(B \otimes Q, 2)$. This is the inductive step having done X_n. Inductively I prove that the cohomology of the space and the original one tensor $\mathbb{Q}$ are the same. You will use the cohomology class on the left to build the cross section on the right. So now we have cohomology in degree three with coefficients in B. Then rationally this lives in $\wedge(x_n) \otimes B \otimes \mathbb{Q}$. That's the obstruction from the Posnikov system but viewed over $\mathbb{Q}$.

You get a map of fibrations, there's a natural map of the bases and the fibers are natural. We'll see in a second it's an isomorphism on the fiber; it's an isomorphism of the base, it's a map of video games and it's an isomorphism at E_2 so it'll be an isomorphism of the total space. This just uses the naturality of the obstruction. So we can, for $K(C,n)$, map C to itself by 2, 3, 4, et cetera, and then the direct limit will have $\pi_n(K(C,n) \to^2 K(C,n) \to \dots)$ equal to $C \otimes \mathbb{Q}$. Any map of a sphere into this infinite mapping torus goes into a compact part of this space, which collapses down to one of these $K(C,n)$ factors so you can prove this isomorphism directly. The homotopy groups just get tensored with $\mathbb{Q}$. In the circle and $\mathbb{CP}^\infty$ you can see this, how do you see it exactly for $K(\mathbb{Z},n)$?

Let's see, I'm going to be using this argemunt again two or three times, let me wait until I get there, so to speak. So one idea here, we will know things about $K(\mathbb{Z},n)$ by induction on n by using the fibration of $*$ over X with fiber ΩX. We know the first two steps, the circle and $\mathbb{CP}^\infty$, and that shows the pattern.

What are some things we're going to want to know?

- $K(A,n) \to K(A \otimes \mathbb{Q}, n)$ is an isomorphism on cohomology with coefficients in $\mathbb{Q}$.
- This cohomology is just $\wedge^n(A \otimes Q)$, the free graded commutative algebra on a graded space.

Let's see, so far what I've shown is that, well, we need to develop an apparatus to make this easy. Given one, we inductively build $(X_{n+1})_{\mathbb{Q}}$ by induction. We have

$$\begin{tikzcd}
K(C,n+1) \arrow[r] & X_{n+1} \arrow[d, shift left] & ?.> \arrow[d, shift right] & K(C\otimes\mathbb{Q},n+1) \arrow[l, dotted] \\
& X_n \arrow[u, shift left, "\theta\in H^{n+2}(X_n,C)"] \arrow[r] & (X_n)_{\mathbb{Q}} \arrow[u, shift right, "\theta_Q\in H^{n+2}((X_n)_Q,C\otimes\mathbb{Q})"'] &
\end{tikzcd}$$

and by moving the obstruction over, we can take the coefficient homomorphism to rationalize the fiber first, and only then use the same cohomology class lifted over the rationalized base.

Then you will get all the same video games. The existence of this video game (the Serre spectral sequence), which is natural, allows you to get the fact, generalizing item one, by induction, that the cohomology over $\mathbb{Q}$ is an isomorphism on the total space. This was all to move up to the next step of the induction.

A cohomology class gives you a fibration of $K(\pi, n)$ and on the other hand the fibration gives you a cohomology class, which is a principal fibration of $K(\pi, n)$ fibering over $K(\pi, n+1)$. A principal fibration is trivial if there's a cross section, so this first obstruction is the only one.

Recall that you can see $H^{n+1}(X, *) \iff [X, K(*, n+1)]$. You can extend a sphere map because it's a cocycle and then it extends over higher cells; the other direction is the obstruction to homotoping a map to the trivial map. Any space you're mapping into is the base of the loop bundle. These kind of fibrations are what's appearing in the Posnikov system. To go from one to another, given these simple fibers, takes only one obstruction. You can tensor with any flat ring.

If you're mapping finite dimensional systems into X you're fine. Anything you really want to do finitely, you can do this way. The skeletons of X themselves are finite dimensional.

The beginning of the proof, the base case, says that it induces an isomorphism on rational homotopy, thence rational homology, thence rational cohomology.

Let me assume 1 and use the argument that proves it in a slightly different form. We'll recover the first one. The way you prove two is sort of the same argument. Assume we have commutative cochains over $\mathbb{Q}$. I want this, to each space I want to assign what are called these. They should compute the cohomology and be geometric, i.e., give you a short exact sequence $0 \to C^k(X, A) \to C^k(X) \to C^k(A) \to 0$. So commutative means that you get a commutative ring. I haven't said that cohomology is a ring. The diagonal map $X \to X \times X$ gives a map $H(X \times X) \cong \oplus H^i(X) \otimes H^j(X) \to H^{i+j}(X)$. A cellular approximation to the diagonal is not commutative. Steenrod flipped these things over, divided by the symmetric group appropriately, and found cochain operations. The question was asked whether we could find one that was still geometric and has a ring structure over $\mathbb{Q}$. This was solved by Quillen in 1968 by Quillen with rational homotopy theory. I noticed in 1970 that you can find another solution to this in Whitney, using differential forms. There are, let's assume we have commutative cochains; by the way, there's an associative and noncommutative multiplication where you just basically, if a simplex is a bunch of symbols, you, the dual map on homology takes $(a_0, \ldots, a_k) \to \sum_k (a_0, \ldots, a_k) \otimes (a_{k+1}, \ldots a_n)$. Some famous mathematicians symmetrized this but it destroyed associativity. All the solutions I know about are infinite dimensional. This is an interesting problem.

Anyway, so I'm sweeping a whole lot of history of topology under the rug, but Whitney had these cochains in the late forties. I got credit for it, but it's due to Whitney. It's okay, you don't always get credit for the things you do, and sometimes you get credit for things you don't do. The cohomology of the building blocks are just the free algebra on the homotopy. So a space will be to make a video game out of the homotopy groups. I'll just make one thing for the video game instead of a bunch of things.

Take something like $K(\mathbb{Q}, 9)$ as fibers over $K(\mathbb{Q}, 10)$. Here's an algebraic model for this. It's supposed to be the free commutative algebra on one generator in degree x in degree ten and one y in degree 9. This is the same thing as $(\wedge(x)) \otimes (\wedge(y))$. This would be the start of the video game. You have $x, x^2, \ldots,$ and $y, yx^2, yx^3, \ldots$ Usually you would write as a spectral sequence,

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and you'd write the other groups down. The differential says $dy = x$ and $dx = 0$. Then you extend d to be a derivation. Then $d(x^n) = 0$ is $d(yx^n) = x^{n+1}$.

We will encode the $\mathbb{Q}$-Postnikov system into an algebraic model, the minimal model of this form, which is $\wedge(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_l)$ with $d^2 = 0$, d a derivation and d on each generator corresponding to the θs. That will be given by the obstructions, that is. There will be a map into forms inducing an isomorphism on cohomology. This will exactly encode the $\mathbb{Q}$-Postnikov system. So the n stage of generators will encode X_n.

This fibration is encoded, if you apply cochains your arrows go the other way. We have one more group and the model of the $n+1$ stage will add another string of generators for the fiber, and the base just sits in here. When you mod out by them you just get the fiber. The fiber has no d. That has no θ, any time you have a free algebra. For example, the infinite Grassmanians are trivial tensor $\mathbb{Q}$. This picture maps into cochains.

This is so easy in spirit. Let me just say, suppose you were just adding one $\mathbb{Z}$, one $\mathbb{Q}$. There's just a $K(\mathbb{Q}, n+1)$. So you have a class that describes this thing. We've built our model up this far. Now we have this class θ, which is represented by a form. If you pull the obstruction up it goes to zero. So you make your model by killing it. In the model it's given by a polynomial. So you just add a new variable with dt equal to this polynomial.

This is free and commutes with d so you can extend it on generators to a map of dg-algebras. Then it's an isomorphism by the Serre spectral sequence and you can go on to the next stage.

The thing I was talking about at the beginning of last semester, Lie infinity algebras, are related to this, but I'll maybe talk about that later.