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1 Algebraic string operations

This is joint work with Thomas Traedler. I'm going to start with defining

Definition 1 *A strong Frobenius algebra (A, U) is a graded associative algebra A and an element $U \in A \otimes A$ of degree $-d$, written $\sum a_i \otimes b_i$ such that*

1. $\sum a_i \otimes b_i = \sum b_i \otimes a_i$
2. $\sum a_i a \otimes b_i = \sum a_i \otimes ab_i$
3. $\sum aa_i \otimes b_i = \sum a_i \otimes b_i a$

Whenever a thing of degree p passes a thing of degree q a sign $(-1)^{pq}$ accrues. I do not assume nondegeneracy. Now, if you have an inner product $\langle , \rangle : A \times A \to \mathbb{Q}$ such that $\langle a, b\rangle = \langle b, a\rangle$, $\langle ab, c\rangle = \langle a, bc\rangle$, $\langle ab, c\rangle = \langle b, ca\rangle$, and $a \mapsto \langle a, \ \rangle$ is an isomorphism, then $\langle , \rangle$ maps via this isomorphism to such a U.

Let X be a Poincaré Duality space of dimension d. Then $\langle a, b\rangle = a \cup b[M]$ and $a, b \in H^\cdot(X)$ are negatively graded, then $U \in H^\cdot(X) \times H^\cdot(X)$ is the Thom class of the diagonal $X \hookrightarrow X \times X$.

Why am I interested in this?

Definition 2 *The cyclic Hochschild complex $CC^\cdot A$ has $CC^k A = \{f : A^{\otimes k+1} \to \mathbb{Q}\}$ where $f(a_0 \otimes \cdots \otimes a_k) = (-1)^t f(a_k \otimes a_0 \otimes \cdots \otimes a_{n-1})$. The differential is $\delta f(a_0, \ldots, a_{k+1}) = f(a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_{k+1}) + f(a_0 a_1 \otimes \cdots \otimes a_{k+1}) + \ldots + f(a_{k+1} \otimes a_0 a_1 \otimes \cdots)$. Then the cyclic cohomology is denoted $CH^\cdot(A)$.*

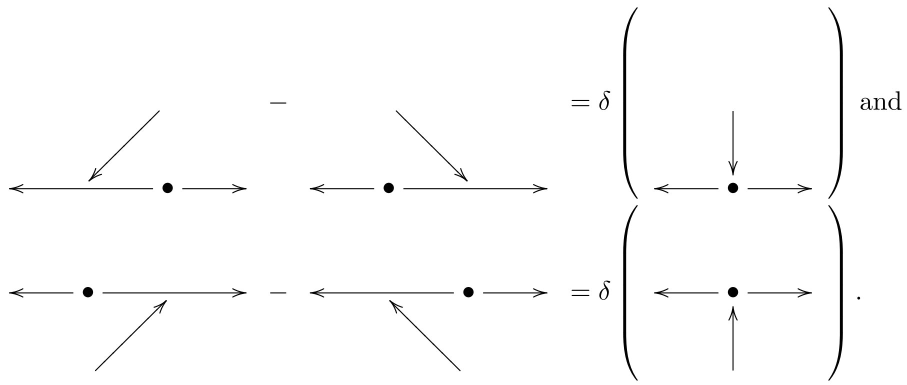
Proposition 1 *if (A, U) is strong Frobenius then $CH^\cdot(A)$ is a Lie algebra after a grading shift, with bracket $[a_1^* \cdots a_n^*, b_1^* \cdots b_m^*] = \sum \circlearrowleft \langle a_k, \delta^{-'} a_i^*\rangle \langle a_\ell, \delta^{-1} b_j^*\rangle a_{i+1}^* \ldots a_n^* b_{j+1}^* \ldots b_m^* b_1^* \ldots b_{j-1}^* a_1^* \ldots a_{i-1}^*$.*

The picture is that you glue a_* to b_* along j in b.

The fun begins when you want to consider something interesting. If you take the Hochschild cyclic cohomology of cohomology you don't get anything interesting. But if you let $A = C^{\cdot}X$ and let $U = \sum a_i \otimes b_i$ be a representative of the Thom class, interesting things happen.

So let's look at the properties you want this to satisfy. This means $\sum a_i a \otimes b_i = \sum a_i \otimes a b_i$. With cup product, multiplying these together is okay by associativity of the cup product. But if the cup product is not commutative, $a a_i b_i = a_i b_i a$ is harder. And this is obstructed. These are simplicial cochains.

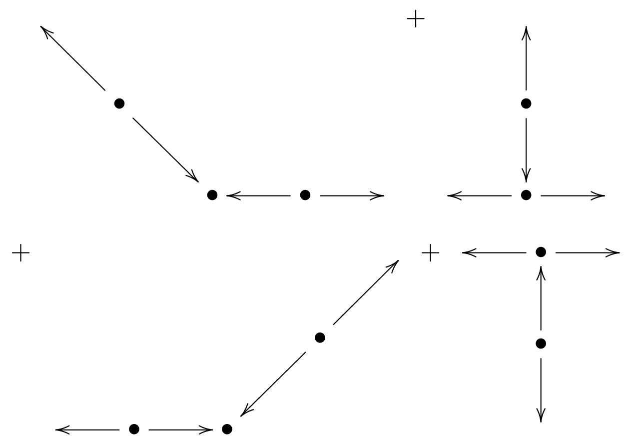
This problem was dealt with in Thomas' thesis. He wrote these graphically.



It makes it look like an A_∞ algebra with an A_∞ coinner product, where you split every vertex in each posisible way.

For an A^∞ algebra with an A_∞ coinner product, $HH^*(A, A^*)$ is a BV algebra.

To get Jacobi, you want to look at the outside of one diagram and put one input on the other's output. So you get a sum



And the cyclic permutations, twelve terms. There is an unfolding for things with more outputs involving δ.

Let $\mathscr{V}_{i_1} \in A \otimes A^{*i_1}$, and in general $\mathscr{V}_{i_1,\ldots,i_k} \in A \otimes A^{*i_1} \otimes \cdots \otimes A \otimes A^{*i_k}$. So a V_k algebra is a complex with these things up to degree k satisfying two properties. You want the correct relations to work. It should be invariant under unfolding. You also want invariance under rotation, for the degree d part by $2\pi/d$.

So it's a complex (A, δ) with the appropriate relation on δ with respect to these elements.

Proposition 2 *Given A a V_3 algebra then $HC^{\cdot}(A)$ is a Lie algebra.*

All of this is the tip of the iceberg. You want to knnow the sturcture of the object at the level of cochains. You want to know the structure of the cyclic Hochschild complex of a V_k algebra. There's a complex $\mathscr{D}\mathscr{G}_k$ and a map from it into $\prod_{r,s\geq 1} Hom(CC^{\cdot}(A)^{\otimes r}, CC^{\cdot}(A)^{\otimes s})$. This is when A is a V_k algebra.

This will respect composition and differential.

I build a vector space first. It is generated by all directed graphs with vertices of type less than or equal to k, so at most k outgoing edges, and a cyclic ordering of the edges at each vertex. Take the vector space generated by all of these objects. Because of the sign issue you want a little bit more. You also want an orientation chosen, by putting the edges in degree one and the vertices in the degree $n(2-d)+d-4$ where n is the type and d is the dimension of U.

This is all graphs, not just trees. The differential will be unfolding so that no $\longrightarrow \bullet \longrightarrow$ are created (loops are okay) Type zero are the inputs. These are ribbon graphs and thicken them. The boundary will be all the outputs. The inputs will be the type zeros. The composition will be, you open up the inputs, you go around the outputs and place them. Let's say you have a one to two and then a two to three.

[Some skipped things]

I am ninety percent sure that V_∞ is a free resolution of strong Frobenius.

[Is there a circle action?]

If you look at the cochain complex and take the Hochschild cohomology, cyclic Hochschild cohomology, you get the equivariant homology of the free loops, $HC^{\cdot}(C^{\cdot}X) = H^{eq}_{\cdot}(\mathscr{L}X)$.

2 Kevin Costello, the B-model partition function

I'm going to start with a bunch of definitions. $M_\chi(n)$ is the moduli space of Riemann surfaces Σ with $\chi(\Sigma) = \chi$ and $\delta\Sigma = \amalg^n S^1$. (explicit homeomorphism)

Let's look at the singular chains $C_*(M_\chi(n) \to^{G_{ij}} C_*(M_\chi(n-2))$. This glues i to j. There's also the operators D_i of degree one by taking the circle boundary component and moving it around.

Is that clear to everybody? These operations satisfy a bunch of axioms, including

$D_i^2 = 0, [D_i, D_j] = [D_i, d] = 0$. We want some kind of field theory for this chain complex. What does that mean? It's going to be a chain complex V with a symmetric pairing $\langle, \ rangle$ which will correspond to the gluing maps, and a degree one operator D of square zero. Um, so the most important part of the data is that there is a chain map from this guy $(M_\chi(n)) \to V^{\otimes n}$ which satisfies the nice axioms you might think. You want gluing of surfaces to correspond to the inner product and the circle actions should correspond. It should be S_n-invariant, and the disjoint union of surfaces should correspond to the tensor product.

You may ask, do we ever actually get such a structure? The answer is, not quite, but essentially. There are some technical modifications. It's very similar to what Mahmoud was saying in the last talk.

Theorem 1 *If A is an A_∞ algebra which is Frobenius (there is a nondegenerate pairing which when coupled with higher tensors is cyclically invariant) then there exists a structure very like this, where $H_*V = HH_*(A)$ and $D : V \circlearrowright$ is Connes' operator.*

This is like the TQFT picture, but that would just be H_0. The circle operator is like the one-chain. So the problem I'd like to consider is, construct some kind of "partition function" for this field theory. Whatever the partition function is, you'd like it to be something like, an integral over the space of surfaces or over Deligne Mumford space. You'd like to map the fundamental class of the Deligne Mumford moduli space in.

But you don't have a circle action in the compactified version. I'll write down an approximation to the fundamental class and we'll use that instead. Let $M_{\chi,n}$ be the quatient of $M_\chi(n)$ by the torus and S_n. This is homotopy equivalent to the space of Riemann surfaces with n punctures. This is what will be compactified to get Deligne Mumford space.

Let $\Gamma_{\chi,n} \subset M_{\chi,n}$, which is just the set of surfaces with their canonical hyperbolic metric and geodesic boundary, all boundary components are of length ϵ, and there are no closed geodesics inside the surface of length less than ϵ.

What can we say about this? It's compact and it's some kind of apporoximation to the fundamental class of Deligne Mumford space. As ϵ tends to zero, this just becomes a node. The problem, when you think of this as a chain, is that it's not closed. It has boundary, it satisfies the following equation $d\Gamma_{\chi,n} + \Delta\Gamma_{\chi,n+2} = 0$. Its boundary is the sum over the ways of gluing two things together. Δ is an operator of degree one on chains which sums over ways of gluing two boundaries. But because the boundaries are not parameterized you have an S^1 worth of ways. This is Sen-Zwiebach-Sullivan. The point that this is really the fundamental class was explained to me by Dennis. We have this thing that is supposed to satisfy this equation. So Δ is an operator $C_*(M_{\chi,n}) \to C_{*+1}(M_{\chi,n})$.

This is going to be more like the real blowup of Deligne Mumford along the boundary.

You have the following

Theorem 2 *There exists $S_{\chi,n} \in C_{-3\chi-n}$ which satisfy the equation $dS_{\chi,n} + \Delta S_{\chi,N+2} = 0$. We also have $S_{chi,-3\chi}$ is a point, union of three punctured spheres*

The series $S = \sum \lambda^{-\chi} S_{\chi,n}$ is unique on $d + \Delta$ homology. We have bounds on homological deimensio of $M_{x,n}$ by Harer.

What is $S_{-1,1}$? it's an essentially unique two chain bounding the glued together $-1, 3$ chain.

Pick $S_{-1,1}$ and $S'_{-1,1}$ both satisfying $dS_{-1,1} = \Delta S_{-1,3}$. Then $d(S_{-1,3} - S_{-1} = 0$, and $S_{-1,1} - S'1, -1 = d\eta$.

So S lives in $M_{\chi,n}$ we need to do o couple of things. First we need to deal with the circle action $Sym^n(V_{S^1})$. What is this? It's $t^{-1}V[t^{-1}]$, and differential d+Dt. This computes cyclic cohomolog. We take the image of the chain under ths map This is an element of the symmetric algebra. So $\mathscr{D} = \sum_{\chi,n} \lambda^{-\chi} \phi S_{\chi}, n \in Sym^* V_{S^1}[[\lambda]]$.

So $\mathscr{D}$ is a partition function, $(d + \Delta)\mathscr{D} = 0$ for some operator Δ on $Sym^* V_{S^1}$.

Now $V_{Tate} = V((t))$ a differentil dttD. This computes periodic cyclic homology. The sypmplectic form is $\Omega(V_1 f_1(t), V_2 f_2(t)) = \langle v_1, v_2 \rangle Res\ f_1(-t) f_2(t) dt$

So dttD is skew adjoint with Ω. We have that this V_{Tate} is the diferect sum of Lagrangians. $Sym^* V_{S_1}$ is a Fock space for V_{Tate}

This is a model for the Weyl algebra of $V((t))$. It's the free algebra on $\Omega(a\ a')$.

Theorem 3 *The quantized operator associated to dttD, $\widehat{dt + D}$ is $d + tD + \Delta$ on $Sym^* V_{S^1}$.*

So maybe one way to say Γ is quantized $--$I should probably stop.

Corollary 1 *The partition function $[\mathscr{D}]$ of is in a Gock space for $H_*(v((t))) == HP_*$*(Frobenius A_∞ algebra).

3 Godin

3.1 String Topology a la Chas-Sullivan

Let X be any space and $x \in X$ a point. Consider $\Omega_x X$ to be $\text{map}((S^1, 1), (X, x))$ then you get a product $\Omega_x X \times \Omega_x X \to \Omega_x X$ by composition.

If you have a manifold and the manifold is oriented, then there's an intersection pairing on the chains of the manifold $C_p M \otimes C_q M \to C_{p+q-n} M$.

Then you have a twisted product with the manifold by allowing the basepoint to move. Let $LM = \mathrm{map}(S^1, M)$. These are loops anywhere on M. You can't really compose them. You have an evaluation map

$$\begin{tikzcd} \Omega_x M \arrow[r] & LMv \arrow[d, "e"'] \\ & M \end{tikzcd}$$

So you get the Chas-Sullivan product $H_p LM \otimes H_q LM \to H_{p+q-n} LM$.

3.2 A different point of view, Cohen-Jones

Consider maps from the figure eight K, composable loops. You have $\Psi : Map(K, M) \hookrightarrow LM \times LM$. If you can come up with a shriek map for this embedding you can compose loops.

Proposition 3 *Ψ is a finite codimensional embedding with a nice tubular neighborhood.*

Corollary 2 *You will be able to build a Pontryagin Thom callapse map $LM \times LM \to Thom(\mathrm{map}(K, M))^{ev^*TM}$ and then using the Thom isomorphism $H_* LM \times LM \to H_* \mathrm{map}(K, M) \to H_*(LM)$. You only need to see that two maps agree at one in order to compose them.*

You have the pullback diagram

$$\begin{tikzcd} \mathrm{map}(K, M) \arrow[r, hook, "\psi"] \arrow[d, "ev_1"'] & LM \times LM \arrow[d, "ev_1 \times ev_1"'] \\ M \arrow[r, "\Delta"'] & M \times M \end{tikzcd}$$

So we really have a map of a fat graph into M and we can do this over families of fat graphs, so moduli space of Riemann surfaces. So you should be able to come up with operations parameterized by the homology of moduli space.

3.3 Conjecture

Let $S_{g,p+q}$ have p incoming and q outgoing boundary components. Let $M^{\delta}_{g,p+q}$ be $\pi_0(Diff(S_{g,p+q}; \delta))$, where the boundary is fixed pointwise. You can glue to get $M^{\delta}_{g,p+q} \times M^{\delta}_{h,p+r} \to M^{\delta}_{\ell,p+r}$.

Conjecture 1 *$H_* LM$ is an algebra over $H_* BM^{\delta}_{g,p+q}$. This means for each p, q you get a map $\phi_{p,q} : H_* LM^{\otimes p} \otimes H_* M^{\delta}_{g,p+q} \to H_* LM^{\otimes q}$. which glue properly. Disjoint union of surfaces corresponds to tensor product.*

This is joint work with A. Ramirez. There's a particular model, which I will define, and then I'll show you the construction.

3.4 A model for $BM_{g,p+q}$

Definition 3 *A fat graph is a combinatorial graph with a cyclic ordering of the edges coming into each vertex.*

Here's an example:

$$\begin{array}{c} u \\ /\,|\,\backslash \\ A\,B\,C \\ \backslash\,|\,/ \\ v \end{array}$$

and you can take the orderings from the embedding $\sigma_u = (ACB), \sigma_v = (ABC)$. So these are the spines of the surfaces $S_{g,p+q}$. The cyclic ordering corresponds to the orientation. The boundaries correspond to cycles in the oriented edges. Choosing different orderings can lead to different surfaces.

Consider the following category of fat graphs. The objects are fat graphs Γ, along with $w_i, 1 \leq i \leq p+q$, an oriented list of cycles, disjoint for the p but not the q, partitioned into the incoming (first p) and outgoing (last q), and $\vec{L} = L_i$. This is a one-valenced vertex in w_i.

As an example a hanging edge on each cycle. The morphisms are collapses of disjoint of tree-that preserve the L_i and the ordering.

Theorem 4 *Thurston, Mumford, Harer, Penner, −−*
$|\mathscr{F}_{p+q}| \cong \amalg_g BM_{g,p+q}$. *for* $q \geq 1$

3.5 Construction of the operations

For any $\Gamma \in \mathscr{F}_{p+q}$ we have $\mathrm{map}(\Gamma, M) \hookrightarrow LM^p \times PM^{gE} \leftarrow LM^p \times M^{gE} \to LM^p \times (\mathbb{R}^N)^{gE}$.

The construction has been done more generally. These are not transverse. This is where the Thom stuff comes in. I'll just write down some propositions. The things you want to reverse have finite codimension so you have $LM_+^p \wedge (S^N)^{gE} \to Thom(\overset{\zeta^{\oplus gE}}{LM^p \times M^{gE}})$ where ζ is the normal bundle $M \hookrightarrow \mathbb{R}^n$.

This gives $LM_+^p \wedge (S^N)^{gE} \to Thom(\overset{ev^*\zeta^{\oplus gE}}{LM^p \times PM^{gE}}) \to Thom(\overset{TM^{*n} \oplus ev^*\zeta^{\oplus gE}}{\mathrm{map}(\Gamma, M)}) \to Thom(\overset{TM^{\oplus gH} - TM^{\oplus gE + gV}}{\mathrm{map}(\Gamma, M)})$

For all Γ we have $H_1(\Gamma, \delta_{IN}\Gamma) \otimes TM$.

Proposition 4 *1.* $\Sigma_+^\infty LM_+^p \to Thom(V_\Gamma)$

2. $\varphi : \Gamma \to \tilde{\Gamma}$ *with*

$$\begin{tikzcd}
\Sigma_+^\infty LM_+^p \arrow[r] \arrow[dr] & Thom(V_{\tilde{\Gamma}}) \arrow[d] \\
& Thom(V_\Gamma)
\end{tikzcd}$$

You get $H_*LM^{\otimes p} \to H_*LM^{\otimes q}$ for each Γ in $\mathscr{F}_{p+q}$ and $M_{top}(M) = \{(\Gamma, f : \Gamma \to M)\}$. We should get $\Sigma^\infty LM_+^p \times \mathscr{F}_{p+q} \to Thom(\overset{V}{M^{top}}(M))$

4 Sullivan

All right, I should really call this talk, "Can I learn some algebra at this conference and apply it to contact geometry?" Toward the end I'll get to some speculation, get to some joint work with Dennis. The more solidly based stuff is joint work with [unintelligible]. Let's start with definitions.

A contact manifold (M, ξ) is a pair. M is a $2n+1$-dimensional manifold and ξ is a $2n$-dimensional maximally nonintegrable. So $\xi = \ker \alpha$ and $\alpha \wedge (d\alpha)^n \neq 0$.

We say $S \subset M$ is a Legendrian submanifold if

1. The tangent bundle sits inside the distribution $TL \subset \xi$
2. $\dim\ L = n$.

The basic example, the one that people start out by drawing, M is $\mathbb{R}^3$, and $\ker \alpha$ is $dz - ydx$.

[picture]

This is the standard three dimensional example. Any contact manifold locally looks like this kind of thing, like a symplectic thing. Examples of Legendrian submanifolds, let's see, down here you might have an immersed curve, and up here a front. Being Legendrian means satisfying this equation locally. The y coordinate can be gotten as $\frac{partialz}{\partial x}$. and so on. You can do this with the one-jet space. Let $(M, \xi) = (T^*N \times \mathbb{R}$ and $\ker = \{dz - pdq\}$.

There's the question, what are the Legendrian knots in $\mathbb{R}^3$. Recently there's been this way of studying these using holomorphic disks. Most of what I'll talk about is well-known, but then there's a bunch of algebra that is not known.

4.1 Legendrian Contact Homology

This is "level one." Contact geometers never think about not taking homology. In this contact manifold, the one of interest is this symplectic manifold. Project out the z direction and get $(T^*N, dp \wedge dq)$. Then take the Legendrian and project it down. If L is Legendrian, then its projection $\Lambda \in T^*N$ is an immersion is a Lagrangian immersion. This means the dimension is half, $\dim \Lambda = n$, $dp \wedge dq|_{T\Lambda} = 0$ and the integral of pdq around any smooth loop is zero.

Generically we can assume that self-intersections are transverse. Then double points of Λ be $DP(\Lambda)$. We need an almost complex structure J, an endomorphism of the tangent bundle $T(T^*M))$ with $J^2 = -Id$ and $\omega(\bullet, J\bullet) > 0$. This is called an ω-tame almost complex structure. This choice exists in a contractible set.

What else do we need?

Let $D_k \subset \mathbb{C}$ be a disk with k marked points. You can give this a conformal structure.

Look at $a, b_1, \ldots, b_{k-1} \in DP(\Lambda)$ Then $\mathscr{M}(a, b_1, \ldots, b_{k-1}) = \{u : (D_k, \delta D_k) \to T^*N, \Lambda)|du + JduJ = 0\}$. I think of the punctures as mapping to corners in the disk.

Then I mod out by conformal reparameterization. This is my moduli space. Unstable nodal disks are now stable nodal disks. The algebra will have to account for this. Where is the algebra?

Let $\mathscr{A} = \mathscr{A}(L)$ be an associative noncommutative unital algebra freely generated by $DP(\Lambda)$. There is the $|a|$-grading ("Maslov index"). Finally the interesting thing is the differential, but I need to make one statement.

Theorem 5 *–, [unintelligible]*
This is obvious to believe but technical to prove. For generic J or L, this moduli space gets to be called a manifold and it's compact in the sense of Gromov. This means you can allow, if you go to the boundary, codimension one and two phenomena, corners and corners of corners, and there's a dimension formula, $|a| - \sum |b_i| - 1$.

Now we can define a differential. This is, look at all moduli space of dimension zero. Because this is compact, $dx = \sum_{\dim\ \mathscr{M}=0} \pm\mathscr{M}(x, y_1, \ldots, y_k)y_1, \ldots, y_k$ is finite.

So, all right. I should,

Theorem 6 *(Same)*
$d^2 = 0$. If L and L' are Legendrian isotopic then the algebra for L and the differential $d_{L,J}$ is stable tame isomorphic to $(\mathscr{A}(L'), d_{L',J'})$. All you have to do is add x and dx and change bases.

I should say, we didn't come up with this algebraic structure, that's due to Chakonov and Eliashberg. They were thinking about knots, and mod two.

Okay, so that's contact homology. I should draw the picture of $d^2 = 0$. Let's look at $d^2 a = 0$. Because it's not stable you need a unit.

Okay, fifteen minutes. Contact geometers are somewhat crude. They want some applications. There are many classification results. These results point toward being more geometric. There's no topology in two in five and three in seven. But we have infinite families for all dimensions greater than 3 for the top manifold. Spheres, tori, Riemann surfaces.

There's also kind of a dynamics application. This is kind of unfortunate, this technical hypothesis. Assume the dga can be linearized. Then if you want to count the number of double points. The number of double points is at least half the sum of the Betti numbers, when the cotangent bundle is $\mathbb{R}^{2n}$. Gromov showed that it has to be at least one.

Another result is kind of a homotopy classification. There exist loops of Legendrians that are smoothly contractible but not Legendrian contractible. This goes by constructing a map. You can take any example constructed for knots in [unintelligible]'s thesis and do a spinning construction to generalize it.

More recently it's becoume interesting, this is going back to topology, there's this, let me run through a construction. There's a knot dga, combinatorially defined. Ng has defined this. It's proven to be quite useful. It can detect the unknot. I don't know about it. The motivation comes from a picture Eliashberg told us. A knot in $\mathbb{R}^3$, you can do the conormal construction, looking for cotangent vectors of norm one and which vanish on the tangent bundle of the knot. This is a Legendrian torus in the unit cotangent bundle, that is, $T^*S^2 \times \mathbb{R}$. This combinatorially defined knot homology $HN_*(K)$ is the same thing as the Legendrian contact homology on this conormal. If this were $\mathbb{R}^n$, this would be $S^{n-1} \times \mathbb{R}$.

So, all right, those are some applications of the level one theory. I should also mention the analogous theory. You replace A_∞ with L_∞, which all comes under the guiding of symplectic field theory. This has levels one through three. First you'd want to include more than one positive puncture, that's level two. Then you want to add genus in level three. It's already been applied, I just found this. There are things which are diffeomorphic but not symplectomorphic, lens spaces, and you need level two for that.

We want to repeat for the Legendrian. But if you have more than one positive puncture, and you unfold this, what do you get? Unfortunately you can also glue at an edge. This was not studied in level one because there was only one positive puncture in each component. This kind of degeneration is unknown. It has at least one in one direction and so on.

Then of course you also have problems with like stable disks.

One final speculation, and then we can have our theory, which people are thinking about more than the level two theory.

The other month we heard a talk where people are trying to link level two symplectic field theory to string topology. the level two relative symplectic field theory of L_K should look like the string topology of K. These are closed and open, respectively.

We'll end it there.

5 John Terilla

I think I want to spend five minutes at philosophy and motivation, then a definition, I'll define a word, quantum background, then a theorem, then an example. So a lot of the motivation for what I'm going to discuss comes from quantum field theory. There you have some data, you have fields, a BV operator Δ which acts on functions on the fields, you have a path integral defined on the kernel of Δ into the ground field k. You have an action S acting on fields, with $\Delta(e^{-S/\hbar}) = 0$. This is the data involved in a quantum field theory. Once you have this data you have observables like $\mathscr{O}$, a function on fields that can be integrated against $e^{-S/\hbar}$. This integral is $\langle \mathscr{O} \rangle$, the correlation function. You thus need $\Delta(\mathscr{O}e^{-S/\hbar}) = 0$, which is the same as $(S, \mathscr{O}) + \hbar\Delta\mathscr{O} = 0$. One thing that occurs is that this correlation function is hard to compute, maybe impossible to compute, maybe undefined.

So I think of thickening a QFT, say $\mathrm{QFT}_0 \subset \mathrm{QFT}_t$. But in this situation, you can then study how this quantum field theory depends on t. So you can look at $\frac{partial}{\partial t}\mathrm{QFT}_t$. The idea is, by deforming the QFT you can get information about it at the origin. So let me, uh,

[Will the derivatives of those undefined things be easier to define?]

Yes. I'm going to make a math definition in a second and we won't have this integral.

The idea is deform the quantum field theory and find out what we can.

Here's the idea of an infinitessimal deformation of an action S. This is $S_t = S + tP$. This has the property that this is an action "mod t^2," meaining $(S, P) + \hbar\Delta P = 0$. The conclusion is that the observables correspond to infinitessimal deformations of the action. In fact it goes beyond this. This extends to equivalence relations. The idea here is that the relation between observables and infinitessimals provides the interface through which QFT and deformation theory interact. You can use deformation theoretic tools to find the correlation functions of the QFT. The correlation functions are determined by an ordinary quantum mechanics over the moduli space of quantum field theories.

Our thinking extends beyond the BV case, which is not particularly general.

This is joint work with J.S. Park and T. Treadler.

Definition 4 *A quantum background $\mathscr{B}$ is a four-tuple $\{\mathscr{P}, m, N, |\varphi\rangle\}$ where $\mathscr{P}$ is a graded noncommutative associative algebra over $k[[\hbar]]$. I want to assume that $\mathscr{P}/\hbar\mathscr{P}$ is commutative. I want to assume it's a free module over $k[[\hbar]]$. Okay. m is an element of $\mathscr{P}^1$ satisfying $m^2 = 0$. N is a graded left $\mathscr{P}$-module which is also free as a $k[[\hbar]]$-module, and $|\varphi\rangle \in N^0$ is annihilated by m, $m|\varphi\rangle = 0$.*

You can think of $|\varphi\rangle$ as a vacuum, an action. There were examples of this in the last talk and in Kevin's talk.

What do you do with this definition? Consider the quantum master equation $me^{-\pi/\hbar}|\varphi\rangle = 0$. This is an equation for π. This will be in $\mathscr{P} \otimes \mathbf{t}$, where this is some kind of a parameter ring, like the maximal ideal of an Artin algebra, something that would make this converge, make everything make sense.

There are negative powers of $\hbar$. I can create a new background $\mathscr{B}_t = (\mathscr{P} \otimes \mathbf{t}, m, N \otimes \mathbf{t} \otimes k((\hbar)), |e^{-\pi/\hbar}\varphi\rangle)$ and $QME(\pi)$ implies $\mathscr{B}_t$ is a background.

We think of this as, mentally, this is an evolution of the vacuum $|\varphi\rangle$ to $|e^{-\pi/\hbar}\varphi\rangle = |\Phi(t)\rangle$.

There's one other important interpretation, you can conjugate the QME by $e^{\pi/\hbar}$, which formally gets rid of the negative powers of $\hbar$. If I call $\mathscr{B}'_t = \mathscr{P} \otimes \mathbf{t}, m^\pi = e^{\pi/\hbar} m e^{-\pi/\hbar}, N \otimes \mathbf{t}, |\varphi\rangle$. "Evolution of m to $m^{\pi(t)}$."

Okay, so that's the quantum master equation. Asking about changing parameter rings brings you to a moduli problem and a theorem, there exists a moduli space for solutions to this equation. That comes from, the QME gives rise to a functon $QFT_{\mathscr{B}}$ from the category of parameter rings to the category of sets, taking background data to the set of solutions to a QME. You make a category of quantum backgrounds in which $\mathscr{B}_t$ and $\mathscr{B}'_t$ are quasiisomorphic and give rise to the same functor.

There exists a universal solution $\Pi \in \mathscr{P} \otimes R$, where R is the universal coefficient ring, to a QME.

There's a notion of equivalence of solutions, which really leads you to prohomotopy representation of this functor.

[The parameter rings have differentials in them?]

Yes. Assume I'm givien a background $\mathscr{B}$ and that the differential in the representing ring R is zero. Then I have the following theorem.

Then I should say, there exists a sheaf $\mathscr{D}$ over the moduli space $\mathscr{M}$ of $k[[\hbar]]$-modules, which, actually, to define this sheaf, look at the universal background $\tilde{\mathscr{B}} = (\mathscr{P} \otimes R, m, N \otimes R \otimes k((\hbar)), |e^{-\pi/\hbar}\rangle)$. As a free R-module this is the source of this sheaf.

Theorem 7 *For every universal solution π to the quantum master equation there exists a flat quantum superconnection $\nabla : \mathscr{D} \to \mathscr{D} \otimes \Omega^{\cdot}(\mathscr{M})$.*

Flat means $\nabla^2 = 0$. Quantum means, well, an ordinary connection would have, this is $\nabla(fs) = \hbar df \cdot s + f\nabla s$. The $\hbar$ is what makes it quantum. Here $f \in R$ and $s \in \mathscr{D}$. Super means that instead of going into Ω^1, it goes into $\Omega^{\cdot}$ So it's $d + A^1 + A^2 + \dots$ The truncated $d + A^1$ is flat in the ordinary sense and is torsion free. So you actually get that you don't need to look for the higher levels. The B-model Frobenius manifold, you don't see these other higher pieces. This is all under the assumption $\delta = 0$. Very interesting activity happens in the other case, but it's not completely clear how to summarize it.

We know how to bypass the obstructions using δ, the functor is still representable, but I don't know how the connection looks. This is like the unobstructed case.

We saw many examples of this today, but the one I prepared, we regard having the quantum background as the starting point, and the flat superconnection is actually the output. This encodes, let me state it, the correlation functions of the original QFT. The goal was to use some deformation theory techniques to compute the correlation functions. The A_1 became the correlation functions, the A_n become higher correlation functions, homotopies of correlation functions.

Okay, so an example.

[I've heard you guys say, to make an analogy, these are like Massey products?]

Here's the example. Say I have a dBV algebra. This is $(V, d, \Delta, \cdot)$. This is a graded vector space, two commuting differentials, a commutative and associative product, and d is a derivation for $\cdot$ while the deviation of Δ from being a derivation of $\cdot$ is a bracket. Assume V is a symmetric algebra and $\cdot$ is the symmetric product.

So $\mathscr{B} = (\mathscr{P}, m, N = V[[\hbar]], |1\rangle)$. Here $\mathscr{P}$ is a Weyl algebra, and $m|v\rangle = |dv + \hbar\Delta v\rangle$. In this example, what does this connection look like? You can identify $\Omega^{\cdot}(\mathscr{M})$ with $\mathscr{D}^*$. Then you have $\nabla : \mathscr{D} \otimes \bigoplus_k \Lambda^k \mathscr{D} \to \mathscr{D}$.

In this case, given Π up to equivalence in $N \otimes R$ we can define a new differential $d + \hbar\Delta + (\Pi, \cdot) = D_\Pi : V \otimes R \to V \otimes R$. Then the homology of this $(D_\Pi)^2 = 0$ is $\mathscr{D}$.

Here $R = S^{\cdot}(H(V[[\hbar]], d + \hbar\Delta))$. That's what R is. Then Π gives a new differential.

I'll just take one minute to kind of explain what the connection means in this case. You have $d + A^1 + A^2 + \cdots$ These have a complicated dependency on $\hbar$ and R. Look at $\hbar$ in separate powers. Set $t = 0$ for simplicity

A^1_0 is the product with the differential. The A^2_0 is a map $\mathscr{D} \otimes \Lambda^2 \mathscr{D} \to \mathscr{D}$ which is the first C_∞ Massey product. This has an ambiguity corresponding to the choice of π as a universal solution. This is the tip of the iceberg.

The first line is very like the rational homotopy theory of $(V, d, \cdot)$. I claim this theory defines these other things, they're not things we've seen before maybe. In the Gromov-Witten theory, they look at $\hbar^k A^1_k$, which gives a Frobenius manifold, but this is the full picture. This is a generalization, but you have to know something about the $\hbar$ dependence. An infinity version of a Frobenius manifold would be a subset of this structure. So, this is the end.

[If you restricted ∇ to the right hand factor, look at that map, that corresponds to an L_∞ structure. Evaluate at $1 \in \mathscr{D}$. That still squares to zero and gives an L_∞ structure. Have you thought about that?]

I can make one remark that uses that phrase, which is that, I talked about $(V, d, \cdot)$. I can also talk about $(V, d, [\,,\,])$. In this case this is formal and so as an L_∞ algebra it's quasiisomorphic to R which is a power series ring, a symmetric algebra of something, so it's a minimal model and Π is a minimal map to the minimal model. The $d \neq 0$ corresponds to an obstruction and you have to glue $(V, d, \cdot)$ to $(V, d, [\,,\ ])$.

The best statement here uses the language of filtered KGAs. If this has special coordinates then Π is a quasiisomorphism from the background data to a dgA which is the minimal model.

6 Scott O. Wilson

The last speaker is Scott Wilson, who will talk about supersymmetric algebra and combinatorial topology. We should thank the organizers for bringing us here and putting it on.

I want to talk about a piece of algebra. I want to list, abstractly a piece of something, an algebra in differential geometry and combinatorial topology. Then I want to do examples, which will justify thinking about it combinatorially.

M is a smooth manifold. I can think of the differential forms. $\Omega(M)$ is a DGA, or I can think of it as a (left) module over itself by multiplication, L_ω. A natural thing to consider on a smooth manifold is a Riemannian metric. Now each of the left multiplication operators have adjoints L_ω^*, which should be called contraction by the vector field dual to the one form. In physics jargon this might be creation and annihilation operators. I have d and with the Riemannian metric I have the adjoint of d. These two together form the signature for a generalized Dirac operator. You can combine Poincaré duality with this and get a Hodge star operator. There's a nice compatibility, left multiplication and contraction being compatible with the Hodge star operator, which we might call Frobenius because $\langle \bigstar(\omega_1 \wedge \omega_2), \eta \rangle = \langle \omega_2, \bigstar(\omega_1 \wedge \eta) \rangle$. This is called $N = 1$ and $N = (1, 1)$ supersymmetry without the star operator and $N = 2$ supersymmetry with.

In topology to do a version of this we should do a higher homotopy version of this. This statement is about being a left module over yourself. A compatibility like this is a Frobenius or Poincaré duality structure. I want to do four examples in combinatorial topology. Some will be trivial, but they will be interesting. My four reasons are

1. Interesting (Lie) algebra.
2. I will use a simplicial complex, so my second thing will be a combinatorial invariant
3. a nice homotopy invariant due to Sullivan, Renitski
4. A renormalization procedure

Let me say one more thing about differential geometry. I want to think about wedging and contracting all at once. This is a story that appears in the construction of Clifford bundles. You take the one-forms acting on themselves. You get a left multiplication operator and its adjoint operator. There's wedging with and contracting with ω. I mean that $L_{\omega \wedge \eta} = L_\omega \circ L_\eta$

by associativity. The difference between the two is not a map of algebras, but the relation it satisfies is the Clifford relation. At a point one would lift to the tensor algebra and mod out by the kerne which is the Clifford relation. That's forming an algebra where you have left multiplication and the adjoint all at once. Here you have an interesting algebra inside the Clifford algebra, $so(n)$.

Let C^k be the simplicial cochains of K. consider and associative product $\cup$ on $(C^\cdot, \delta)$ and the map $a \mapsto L_a$. I want to assume that $C^\cdot$ has an inner product, where simplices of different dimensions are orthogonal. In the presence of the inner product you have a map $C^\cdot \to Hom(C, C)$ given by $a \mapsto L_a^*$. I want to study this algebra for the following space K:

$$a \overset{e}{\text{———}} b$$

My $\cup$ will be $ae = e,\ eb = e, a^2 = a, b^2 = b, ab = 0$. Consider the maps $C^\cdot \overset{L,L^*}{\to} Hom(C, C)$. Then the operators L_e and L_e^* satisfy $[L_e, L_e] = 0, [L_e^*, L_e^*] = 0$ and $[L_e, L_e^*] = h$ is a central element. You might think of this as a Heisenberg algebra in degree one. With the circle we get a central element which is the identity. If we divide the circle into a_i and e_i, you get the operators $L_{e_i}, L_{e_i}^*$ with $[L_{e_i}, L_{e_j}^*] = \delta_{ij} h_i$ and $\sum h_i = id$. There is a locality relation that things commute if they are not near one another.

Let me draw a little picture here. I want to include my differential.
$$\begin{array}{c|cc} L_e^* & [d, L_e^*] & [L_e, L_e^*] \\ d^* & [d, d^*] = \Delta & [d^*, L_e] \\ \hline & d & L_e \end{array}$$
The bottom right commutator is a chain homotopies for how left multiplication commutes with the Laplacian if x is a cocycle. These live in $Hom(C, C)$. The upper left is a chain homotopy for contraction by something exact and the Laplacian.

In a very special case we saw a Lie algebra popping up.

Let me say something about this piece here that I heard a month ago. Suppose we want to deal with connected graphs and we orient everything, so Γ, a graph, is a simplicial complex. I want to choose the standard inner product making the cells orthonormal on C. I want to take the adjoint of the coboundary δ which is the boundary ∂. I want to consider $\delta + \partial$ and square it, as a map $C^0 \to C^0$. This is like a Laplacian; it has eigenvalues, one of which is zero because H^0 of a connected graph is one dimensional. Then the determinant (the product of nonzera eigenvalues) is th e product of the number of vertiecs. with the number of maximal trees in Γ restricted to $C^0 \to C^0$.

For K any simplicial complex you can look at the Laplacian $C^i \to C^i$ and ask what the determinant equals. Roughly, you can say what a maximal spanning subcomplex is, it's of dimension $i + 1$ and supports no cycles. In that is a choice of something in dimension i that supports no cocycles. The determinant is sort of counting the number of these things. There's something about the torsion homology of this. That's another thing one might pursue. Okay.

Let me say something about the third one. I had the Hodge star operator. Well, there's this whole subject of index theory involved in computing indexes of operators and many familiar examples show up in Riemannian geometry. If M is a $4k$ manifold then $*$ maps from i to $4k - i$ In the middle dimension this is nondegenerate, and you can think that $*$ gives a $\mathbb{Z}_2$

grading of the forms. A fact is that if one takes the dimension of the kernel of $d + d^*$ minus the dimension restricted to the odd, the index of this operator with respect to this $\mathbb{Z}_2$ grading with respect to $*$, this is the signature of the manifold.

There's a combinatorial version, a topology version of this due to Sullivan and Renichki. You have a triangulated $4k$-manifold and you consider a cup product on cochains. You can define a Hodge $*$ operator by taking the cup product and evaluating on the fundamental class. Here I want to use a particular commutative cup product.

The signature of a $4k$-manifold can be commuted by coboundary, its adjoint, and $*$. I'm not making completely precise the connection with index theory, but this is a window to doing it. So, uh, the last thing I want to say is that the loop space is not very far off from what I'm discussing. The Hochschild complex can be used to compute the homology of the free loop space. The story of left multiplication is not too far off on the loop space. One thing I'm sort of interested is doing the signature kind of thing in the loop space.

Cochains on a triangulated space should be thought of as piecewise linear forms. There's a collection of theorems, one due to one of our organizers is that with a fine triangulation you can approximate many things. The Laplacian, the Hodge star operator. This is more than making an approximation with convergence results. You want to study things that are not defined on the chains.