Dennis Seminar
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There's absolutely no chalk in here. So,

We have several kinds of fibrations. We have geometric, I'm going to ask some of you to give definitions.

[Definitions of all of them.]

Now today we have algebraic fibrations over $\mathbb{Q}$. This illustrates the idea of making any map into a fibration. The space should be thought of as a differential graded commutative algebra, although what we're going to do would make sense with other kinds of algebra. Replace a space by its differential forms or something. Then a map is a morphism of dgcas. The cells were free generators in the old category. Here we'll want some kind of free objects too. Rather than give the definition, I'm going to start with an arbitrary map and convert it into a geometric fibration. So start with a map $\mathscr{B} \to \mathscr{E}$. Fix it to make it an isomorphism on cohomology by adding "new variables" to $\mathscr{B}$. Start with $\mathscr{B}$, I'm going to fix it by adding stuff to make it an isomorphism on cohomology. This thing here will be the new $\mathscr{E}$. Then the new $\mathscr{E}$ will be $B\otimes$ the free thing on the new variables. d on the new variables will create a model for F. We take the total differential. The new variables by themselves will be a free differential algebra, and that will be F.

You "fix it" by induction. Suppose that $\mathscr{E}$ and $\mathscr{B}$ correspond to spaces, so it's already an isomorphism on degree zero. That's like supposing $\mathscr{E}$ and $\mathscr{B}$ are connected. I'll also assume it's injective on H^1. This is like assuming the fiber is connected. Then add generators in degree one to make it onto in H^1. Now add generators in degree one to make it injective in H^2. Now, there's some more stuff in degree two. We also have new cohomology in degree two, then we have to repeat, this is an infinite process. Take the direct limit and you get it injective on degree two.

So then we add two dimensional generators to make it surjective on H^2. From here on out it's going to be one step in each dimension. In some sense we're creating a huge space over $\mathscr{B}$ that's equivalent to $\mathscr{E}$. So then we have something $\mathscr{B} \otimes \mathscr{F}$, where $\mathscr{F}$ is the free algebra on all the new generators. So we have $\mathscr{B}(--, d)$ as the total space.

So an algebraic fibration of $\mathscr{B} \to \mathscr{E}$ is a pair $(\mathscr{F}, d)$ where F is a free graded algebra and d is a differential on the algebra $\mathscr{B} \otimes \mathscr{F}$ which restricts to the differential of $\mathscr{B}$ and makes $\mathscr{B} \otimes \mathscr{F}$. The choice of these variables, modulo $\mathscr{B}$ is part of the structure. There's a nice bit of formalism here. You want this all to be a derivation, so if you expand a product, the coefficients all have to be derivations of the algebra $\mathscr{F}$. So you can write this as a differential in the fiber d_F plus a matrix $A_b \cdot$ and so you get, from $d^2 = 0$, that $d_F A + A d_F + A^2 = 0$. When you have an endomorphism of two chain complexes $A : F \to F$, the vector space of endomorphisms of various degrees form a complex, where the differential is $[d_F, A]$. Then this is $dA + A \circ A = 0$. So we could call this flat.

This A is called the twisting cochain. Chen knows all about this if you want to know about it. If it weren't there, this would be like the tensor product on the Cartesian product.

Exercise 1 *Suppose you have a fibration with fiber S^2. This is modeled by $\mathscr{B}(x, y)$ where $dx = b$ and $dy = x^2 + b'x + b''$. You use $d^2 = 0$ to deduce that you can assume that $dx = 0$ and $dy = x + b$. Then b is in degree four and $db = 0$. You get a characteristic class in degree four. In the appropriate setting this is the first Pontryagin class. With fiber $\mathbb{CP}^n$ you get the Chern classes starting with the second one. This fits with the ideal that this is $PGl(n, \mathbb{C})$ and we've got rid of the first class.*