Dennis Seminar
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It's nice to be back here again, especially for this happy occasion. I have something to say about the prize, I don't really like the "Lifetime Achievement" thing. But Dennis told me it was the achievement so far. That's the Dennis I know.

I want to talk about quantum algebraic fibrations.

I will start with a metaphor. Quantum theory is typically obtained by quantization of a classical object. To get a quantum thing you do something called quantization, whatever that is.

But we believe the classical to be a result of the quantum setting. In a certain sense I always think of, what is the mathematical analogue for this. If I put the quantum mathematics in this picture, a good classical math would be the result of a quantization. That's my metaphor, my assertion.

So first I'll introduce the notion of something I call dequantization of a dgA. Here what I mean by a dgA, that would be a graded space C with d and $\cdot$ and d a derivation of the product. I will only assume associativity. The grading will be $\mathbb{Z}$-grading. This is over k. I will define the notion of dequantization. One of the salient features is the derivation of the product, $d(a\cdot b) = (da)\cdot b + (-1)^a a\cdot db$ and $d^2 = 0$. We can take cohomology and the product descends to H. So on the homology we have a minimal A_∞ structure, a collection of the higher Massey products.

Now I will define an "anti"-dgA. Suppose C is graded, you have an operator called K of degree one and a product. We have the Abelian group structure and a product $\cdot$ over k. I assume no associativity or other restraints. So we have $C^i\otimes C^j\to C^{i+j}$ and $K^2=0$. If I assume additional properties, I get a dGA.

This is good dirt to plant things in.

We want to sort of compare, let's define $\nu_2 : C^i\otimes C^j \to C^{i+j+1}$. Say $\nu_2(a,b) = K(a\cdot b) - (Ka)\cdot b - a(Kb)$, with the appropriate signs.

Now K is a derivation of μ_2 without using any other properties: $K(\nu_2(a,b)) = \nu_2(Ka,b) \pm \nu_2(a,Kb)$.

So if $\nu_2(\nu_2(a,b),c) - \nu_2(a,\nu_2(b,c)) = 0$ you would call this associative, but it is only homotopy associative, you have that this difference is $K(\nu_3(a,b.c)) + \nu_3(Ka,b,c) + \nu_3(a,Kb,c) + \nu_3(a,b,Kc)$ where $\nu_3(a,b,c) = \nu_2(a,b\cdot c) - \nu_2(a,b)\cdot c$.

There are other choices you can make for ν_3 but in the associative world, where we're going, they are homotopic.

The statement is, if we have $(C,K,\cdot)$, then this induces an A_∞ structure (C,ν_i) with $\nu_n : C^{\otimes n} \to C$ which all have degree one.

We have $\nu_m(a_1,\dots,a_m) = \nu_{m-1}(a_1,\dots,a_{m-1}a_m) - \nu_{m-1}(a_1,\dots,a_{m-1})\cdot a_m$.

A morphism will be f_1, f_2 which send $C \to C'$ and $C \otimes C \to C'$ along with the properties f_1 is a morphism of complexes, forgetting the product, both are linear over k, and $f_1(a\cdot b) = f_1(a) \cdot' f_1(b) + f_2(a,b)$. The f_2 is not necessary.

I will make another stantement, theorem, lemma. If we have (C,ν_i) and (C',ν_i') then the morphisms here induce morphisms of the A_∞ structures.

If you consider the left hand site, taking K' of it. Because f_1 commutes with K I get $f_1(K(a,b)) = f_1((Ka)b) + f_1(a(Kb)) + f_1(\nu_2(a,b)) = K'f_1(a) \cdot' f_1(b) + f_1(a) \cdot' K'f_1(b) + f_2(Ka,b) + f_2(a,Kb)$. On the RHS it is $\nu_2'(f_1(a),f_1(b)) + K'f_2(a,b)$ plus other turms to cancel the beginning of the left hand side.

So then I get the first relations. The rest is the boring part and I just skip it.

It's also possible to get an L_∞ structure out of this.

[Suppose this chain map is a quasiisomorphism. Does that agree with the structure coming over via dot?]

Obstructions of obstructions vanish. Associativity of the A_∞ structure is nice, it's the deviation from having structure. You can get an L_∞ structure. You can have a noncommutative version.

Now a third definition. Before I do this, I only want to talk about the K where the dot product is associative. There are various other choices for the homotopy structure, but if dot is associative they are homotopy equivalent.

So we can talk about a filtered KGA over $k[[\bar{h}]]$. This is a KGA over $C, K, \cdot$ with additional properties. ν_n is divisible by $\bar{h}^{n-1}$.

First of all $K^2 = 0$. Then $K(ab) - (Ka)b - a(Kb) = \bar{h}\mu_2(a,b)$. When you define the further μ they have additional $\bar{h}$.

Now things will get interesting. A corollary of this guy is that $\mu_1 = K, \mu_i$, define a usual A_∞ structure.

In a certain sense, it's clear that what this picture says, the K being a derivation of the product is a violation of $\bar{h}$. So if you kill $\bar{h}$ you get a differential graded alegbra. So this is a quantization of a dgA. I can get, at the homology level, μ_2 is K-closed, not exact. This is a little bit more structure.

Choosing this definition of quantization has some physical meaning.

A morphism of filtered KGAs should preserve the filtration. It's like f_1 and f_2 as before but $f_2 \equiv 0 \mod (\bar{h}^{n-1})$.

Let g_n be defined by $\bar{h}^{n-1}g_n = f_n$. So $g_1(ab) = g_1(a) \cdot' g_1(b) + \bar{h}g_2(a,b)$.

Then (g_i) induce an A_∞ morphism of this new guy. From now on I want to live in the world of filtered associative dgA. In certain senses you see this is sort of a quantization of morphisms.

Those g_n can be expanded in terms of $\bar{h}$.

Now I can define a dequantization.

If I think that homology with the underlying product depends only on the class, in this case that doesn't hold, it depends on the representative.

Now let me define dequantization. First I want to regard a dgA as a filtered KGA over $k[[\bar{h}]]$.

From this I think there's an unambiguous way of extending $(C, K = d, \cdot)_{k[[\bar{h}]]}$. Then the dequantization of a dgA is a filtered KGA C' over $k[[\bar{h}]]$ together with a quasiisomorphism $(C, K = d, \cdot) \to (C', K', \cdot')$.

I will answer the question, a physical answer. I call this dequantization. Initially we have a classical object. There is a way of quantizing these, you find a filtered KGA which gives this in the classical limit.

But if I know every function completely, I know this dgA as a result of something else. I didn't tell you how to do that. You started with a quantum field theory and this is the perturbation around that. Any quasiisomorphism will do. The quasiisomorphism as a KGA is the same as for the A_∞ structure. You have to be careful about this, it has a quasi-isomorphism in it.

When you do the Feynmann algorithm, you have a classical construction, it's the perturbative expansion of the real quantum theory around a classical solution. This gives a complete descripiton of all correlation functions. In certain other senses, I sort of regard this as a minimal model of a quantum object.

I need to describe how to get to the ordinary object to the quantum object.

It's easy to say they are quasiisomorphic, but it's hard to construct them. It's easy to say, "let's invert these" but that's not really easy. You get a quasiisomorphism sometimes and you don't know what the thing is. Dennis' minimal model comes with a quasiisomorphism and freeness. It means you can do your computation freely and constructs the quasiisomomorphism.

I think we should take a break. Let's break until five minutes to four. That was very good, all understandable.

[Break]

Okay, let's start again. Maybe the language is incorrect.