Dennis Seminar
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[How do you go back to the Postnikov system from the forms algebra?]

Assume the space is simply connected, otherwise it's like the Hobbit in the Lord of the Rings, possibly an infinite game, because of the action of π_1. But the statement in the simply connected case is, each space X has a model $\wedge(--,--,--;\ d)$ which maps onto forms, a rational isomorphism. These will turn out to be the homotopy groups, these $--$. This thing is in an appropriate sense complete. There's an isomorphism to make things homotopy commutative, a map of DGAs. I didn't prove this. You define t, dt and build your homotopy algebraically, using obstruction theory on the level of algebras. Okay, so in particular each X_n in the Postnikov system has a model. The theorem is:

Theorem 1 *The model of X_n is just the model of X up to n.*

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Suppose you have a couple of stages of the model, and then you add the third stage. The pullback on forms of the Postnikov stage n to $n+1$ is just the inclusion, and then the pullback to the fiber $K(\pi_{n+1}, n+1)$ is just modding out by the lower ones.

If you have $K(A,n)$, the forms on it, by the Hurewicz theorem, $H^n = Hom(A, \mathbb{Q})$. This maps into $\Omega^*(K(A,n))$ as a free algebra on this.

Say $K(\mathbb{Q},3) \to \wedge(--,--,\bullet)$. Now I'll define the transgression. This word was introduced when they only had the cohomology version, no chain version. This $\bullet$, x will be put in in degree four. The first obstruction to cross section is in $H^5(X_3, \pi_4)$, that is, $H^5(X_3, \mathbb{Q})$. This is represented by an element of $\wedge(--,--)$. So you're introducing x to kill this. That is, $dx = p(--,--)$ with $dp = 0$ and p representing $\mathscr{O}$ the obstruction.

Notice that dx is made out of the variables from the base. If we restrict to the fiber, dx becomes zero and x is closed. At the level of forms you have a five form on B, ω_5. Then this is $d\eta_4 = \pi^*(\omega_5)$. A fibration is called totally transgressive if you can do this in this way, if a basis in the fiber extend a non-closed form on the total space, but with d on them they come from the base.

On the fiber these are closed, on the total space they are not closed, but d of them comes from the base. The language is used to describe this, but you don't need the language, you have the picture. That's why I don't know the language, it's not essential to understanding it. When you look at the model, you see a good candidate for the models from the Postnikov system. We prove inductively that if you have the forms on X_2 and the forms on X_3, we can form the maps we want because the things we put in kill the obstructions. So we get

$$\begin{tikzcd} \wedge(--,--) \arrow[r] & \Omega^*(X_3) \\ \wedge(--) \arrow[r] \arrow[u, hook, dotted] & \Omega^*(X_2) \arrow[u, "\text{geometric}"'] \end{tikzcd}$$

Since the obstruction pulls to zero, that's the main point. The obstruction dies in the total space. This uses the tautologous section

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & E \arrow[d] \\ E \arrow[r] \arrow[u, bend left] & B \end{tikzcd}$$

A corollary is sort of a π_* de Rham theorem. $\pi_1(X) = 0$ then $Hom(\pi_i(X), \mathbb{Q})$ is the model up to dimension i relative to the $i-1$ model.

For any $K(--, n)-$fibered fibration, a model of E is a model of B, with adjoined variables $--$ with $d(--)$ equal to the first (and only) obstruction. And it's totally transgressive, the generators transgress to the first obstruction.

You start at the bottom where you know something. You have $\pi_2 \cong H_2$ and then you have the free algebra on H_2. You can pick harmonic forms. Their products might no longer be harmonic. On the higher groups, fix things. They won't be isomorphic, so fix it. There will be quadratic polynomials that satisfy linear relations, so you use degree three to kill these relations and also to add generators. You can do this in either order in low dimensions like this.

So given that we understand the subroutine, we can go further and study any fibration. Suppose we have a base and we have any fiber. Say the fiber is simply connected (abelian π_1) and $\pi_1(B)$ acts trivially on $\pi_*(F)$. There's sort of a Postnikov system for B. There's one for F, and there are fibrations over B with fiber $F_2 = K(,2)$, This is true because the Postnikov system construction is natural. You make a natural choice of attaching cells. You could also do it by obstruction theory. Then you'd have E_3 over E_2 with fiber $K(,3)$. Over a point you have the Postnikov system of the fiber, but you can spread it out.

In general you have your model of B and then you add the groups for the fiber. $(B, --, --, \cdots)$ will be a model in general, cut off at the appropriate dimension. The model of B is sort of in the coefficient ring now. This also gives you a picture of fibrations.

Let's do an example, how about the Hopf fibration? So what I'm saying is that you can model this fibration by taking a model of S^2, there will be some obstruction $\omega \in H^2(S^2)$ and then add y with $dy = \omega$. For the product you would have $dy = 0$. The y is the circle. One generator y in degree one with square zero, this is an exterior algebra.

Suppose x is the nonzero element. Remember the model for S^2 is $\wedge(x, y; d)$ with $dy = x^2$. This is S^2. Now I add another generator u with $du = x$. This is a model for the total space, if it's nontrivial. Note that this is isomorphic to the algebra generated by $\wedge(x, u) \otimes \wedge(\bar{y} = y - xu)$. This is closed. So you get an untwisting to the middle model of the total space. This is a special case, any free commutative dga is equivalent to the product of contractible pairs and minimal models. The point is that it is with the d that this is true. The second part is well-defined up to isomorphism by the homotopy theory of the left hand side.

Now let's do one where you learn something. Let's analyze two-sphere fibrations. I have a base and fiber the two sphere. Learn things now that the other topologists in this department wouldn't know. One example is the twistor space. If you have a four dimensional Riemannian manifold, you can look at the isomorphisms to $\mathbb{C}^n$, look at the Js. In a four dimensional space this is homotopy type the two sphere. Look at the structures where the J is orthogonal, preserves the inner product. Then you get an honest two sphere. For a six, this is a complex projective three space. We proved this a few summers ago, but the garbage can was on top of us at the end of the proof.

Already for a four-manifold this is a tricky animal. Let's understand it rationally. By the previous discussion there's a model of E of the form (model of B)$, x, y$). with $dx = b_3$ in the model of B and dy in the model of B with x adjoined. It will start out with x^2, it will be $x^2 + b_2 x + b_4$. We need π_1 of B to act triviallf on $\pi_*(S^2)$. This would be true in an orientable manifold. It definitely needs to be orientable to have an almost complex structure.

The general theory says you always have an expansion like this. So b_3 is the first obstruction to cross section to E, which we kill with x. We have $d^2 = 0$ so $db_3 = 0$ and $d^2(y) = 2xb_3 + db_2 x + b_2 b_3 + db_4$. So $db_4 = b_2 b_3$ and $-2b_3 = db_2$. This is rationally trivial. So the first obstruction is torsion.

Now we know that the first obstruction is zero rationally. So $dx = 0$ and $dy = x^2 + b_2 x + b_4$. Now I get $0 = db_2 x + db_4$. So $db_2 = 0$ and $db_4 = 0$. So if you let $\bar{x} = (x - b_2/2)$ then you have $d\bar{x} = 0$ and $dy = \bar{x}^2 + b_4$. So it's determined by $H^4(\text{base}, \mathbb{Q})$. If you take an $\mathbb{R}^3$ bundle, this is the first Pontryagin class. That's sort of how you can use it.

I will have a class this Friday and next Monday.