Physics Seminar
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[Maybe we should begin, whoever wants to join us later should...]

This talk is about quiver representations.

We start with a simple question. Given a vector space, what are all the configurations in which two subspaces can sit inside it? Two configurations are the same if there is an automorphism of the parent space taking the two vector spaces and their intersection to the two vector spaces and their intersection.

We're talking about finite dimension, so if I choose a basis for the intersection, I can extend it on each side and then to the whole space, so if these four numbers are the same, we can say these are isomorphic.

So as a picture we can write this $V_1 \longrightarrow V \longleftarrow V_2$. So now what if I have three or four subspaces? That is why we are interested in quivers.

A quiver is just a connected graph. More formally, $Q = \{Q_0, Q_1\}$, with two maps $t : Q_1 \to Q_0$ and $h : Q_1 \to Q_0$.

A representation of a quiver is as follows: fix a background field k. Assume that this is algebraically closed and characteristic zero. Put a finite dimensional vector space in each vertex and a linear map corresponding to each arrow. V is a representation of Q if $\{V(x)|x \in Q_0\}$ are vector spaces and $V(a) : V(ta) \to V(ha)$ is a map.

All these definitions are very simple. There's nothing fancy going on.

Now we want to define morphisms between representations of quivers.

Let's look at an example. Say I have the following two quivers:

$$\begin{tikzcd} V_1 \arrow[r, "f"] & V_2 \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & V_3 \end{tikzcd}$$

$$\begin{tikzcd} W_1 \arrow[r, "f'"] & W_2 \arrow[r, bend left, "g'"] \arrow[r, bend right, "h'"'] & W_3 \end{tikzcd}$$

Then a morphism will be a collection of maps $V_i \to W_i$ making the diagram commute. In general, $Morphism(V, W) = \{\Phi(x) : V(x) \to W(x)\}$ so that

$$\begin{tikzcd} V(ta) \arrow[r, "V(a)"] \arrow[d, "\Phi(ta)"'] & V(ha) \arrow[d, "\Phi(ha)"'] \\ W(ta) \arrow[r, "W(a)"] & W(ha) \end{tikzcd}$$

commutes.

A subrepresentation is as follows: let V be a representation of Q; then W is a subrepresentation of V if $W(x) \subset V(x)$ and $W(a) : W(ta) \to W(ha)$ is the restriction of $V(a)$. A representation always has two subrepresentations: itself, and the zero representation where $W(x) = 0$ and the maps are zero.

If a representation of Q does not have any other subrepresentations, it is called simple or irreducible.

Let me give an example.

$$\begin{tikzcd} \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

What are all the irreducible representations of this quiver? You cannot have dimension greater than one, but after further thought you find that the only irreducible representations are

$$k \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow k \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow k$$

This is true for any quiver which does not have a cycle, k in any location.

So given a morphism between two quiver representations, we can take the kernel, and both the kernel and quotient are well-defined.

The direct sum of two representations, we can guess what that is. Take the direct sum of each vector space and the corresponding map. That's a categorical biproduct. If you can't break up a representation into a nontrivial direct sum then it is indecomposable. Obviously, irreducible implies indecomposable. However, the following are reducible but indecomposable.

$$k \xrightarrow{1} k \longrightarrow 0$$

$$0 \longrightarrow k \xrightarrow{1} k$$

$$k \xrightarrow{1} k \xrightarrow{1} k$$

For the first case, one can only break this up into two pieces one way, but then the map in the quiver representation direct sum would be zero.

So this makes an Abelian category. Now given a quiver Q you can consider $Rep_k(Q)$, which is now Abelian. The zero morphism is the zero representation.

Given a representation you can break up into indecomposable representations. Is this unique?

Given an Abelian category, if every object can be written as a finite direct sum of indecomposable objects and $Mor(A, A)$ is a local ring, then this decomposition is unique.

This is true for this category, so we know that everything can be written uniquely as a direct sum of indecomposables.

What about a quiver that looks like a circle, with an arrow pointing to itself? This is a vector space V and a map f to itself. This map can be put in Jordan normal form, since k is algebraically closed. A representation will be indecomposable if it has only one Jordan block. So in this case we have infinitely many, since we can choose any λ_i in the field and any size for the block. So we parameterize this by $\mathbb{N}$ and k. In particular this is infinitely many. However, I have at most a finite number of dimensions of parameters. So this is called tame type.

Now let's consider all quivers with finitely many indecomposables. Such quivers are called finite type.

Now we ask three questions:

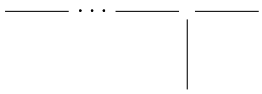
1. Which quivers are finite type?
2. Given a finite type quiver, how many indecomposable representations do you have?
3. Find all the indecomposables.

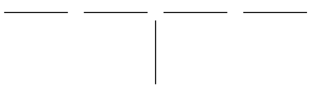
These are the three questions. We have a complete answer for all of these questions.

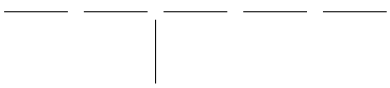
Theorem 1 *Gabriel's Theorem*

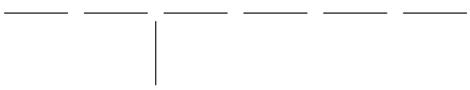
1. *Let $\hat{Q}$ be Q as an undirected graph. A connected quiver Q is finite type if and only if $\hat{Q}$ is one of the following:*

(a) A_n :

(b) D_n : 

(c) E_6 : 

(d) E_7 : 

(e) E_8 : 

2. *Given a finite type quiver, there is a one to one correspondence between positive roots and indecomposable representations.*

   So A_n has $n(n+1)/2$; D_n has $n^2 - n$, E_6 has 36, E_7 63 and E_8 120.

As an example, let's look at D_4.
There are four indecomposables like this:

$$\begin{tikzcd} k \arrow[r] & 0 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

There are three like

$$\begin{tikzcd} k \arrow[r, "1"] & k & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

There are three like

$$\begin{tikzcd} k \arrow[r, "1"] & k & k \arrow[l, "1"'] \\ & 0 \arrow[u] & \end{tikzcd}$$

Then there are these:

$$\begin{tikzcd} k \arrow[r, "1"] & k & k \arrow[l, "1"'] \\ & k \arrow[u, "1"'] & \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r] & k^2 & k \arrow[l] \\ & k \arrow[u, "1"'] & \end{tikzcd}$$

The maps in this last are embeddings so that any two are linearly independent.

Let's do the example of

$$\begin{tikzcd} k^2 \arrow[r] & k^3 & k^2 \arrow[l] \\ & k^2 \arrow[u] & \end{tikzcd}$$

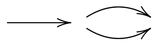
with embeddings along pairs of a basis triple.

This breaks up into

$$\begin{tikzcd} k \arrow[r, "1"] & k & k \arrow[l, "1"'] \\ & 0 \arrow[u] & \end{tikzcd} \oplus \begin{tikzcd} k \arrow[r, "1"] & k & 0 \arrow[l] \\ & 0 \arrow[u, "1"] & \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[r] & k & k \arrow[l, "1"'] \\ & k \arrow[u, "1"] & \end{tikzcd} .$$

I'll talk about the second part next time.

Let's do one direction. Let Q be a finite type quiver. We can take the dimension of each vector space in each vertex; I can always find $\alpha \in \mathbb{N}^{Q_0}$ where $\alpha(x) = \dim\ V(x)$. Then we define the Euler form on $\mathbb{R}^{Q_0}$; This is defined as $\langle \alpha, \beta = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha)$.

For an example consider $1 \longrightarrow 2 \longrightarrow 3$ and  .

So $\langle e_2, e_3 \rangle = 0 - \sum e_2(2)e_3(3) = -1$.

Now define $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$. Then $(e_i, e_j) = 2\delta_{ij} - \#edges_{ij}$.

So in the second case $(e_2, e_3) = \langle e_2, e_3 \rangle + \langle e_3, e_2 \rangle = -2 + 0 = -2$.

We're almost there. Consider these quivers:

$$\begin{tikzcd}[arrows=dash] 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd}$$

$$\begin{tikzcd}[arrows=dash] 1 \arrow[r] & 2 \arrow[r] \arrow[dl] \arrow[dr] & 3 \\ 4 & & 5 \end{tikzcd}$$

Then the matrices for the Euler forms of these quivers are respectivelly

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & -1 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

The first matrix is positive definite (the Cartan matrix of the root system) and the second one isn't.

So a general remark: if I have $\mathbb{R}^n$ and positive definite $(,)$ with $\{v_1, \ldots, v_n\}$ linearly independent, $(v_i, v_j) \leq 0$ for $i \neq j$ and $\frac{4(v_i,v_j)^2}{(v_i,v_i)(v_j,v_j)}$ then the corresponding graph is one of those mentioned, $A_n, D_n,\ E_6,\ E_7$, or E_8.

Let $e_i, 1 \leq i \leq n$ be the vertices of finite type Q We want to show that

1. The Euler form is positive definite
2. (e_i, e_j) is $-\{\#edges\ i \iff j\}$
3. $\frac{4(v_i,v_j)^2}{(v_i,v_i)(v_j,v_j)} = (e_i, e_j)^2 = 0$ or 1.

The only hard part is to show that the form is positive definite; everything else is easy.

I won't show everything, just the idea.

Actually, let me do something else. Fix a dimension vector α. Then $Rep(Q, \alpha)$ is nothing but choosing a linear map so is $\bigoplus_{x \in Q_1} Hom(k^{\alpha(tx)}, k^{\alpha(hx)})$.

Now consider $GL(\alpha) = \prod_{a \in Q_0} GL(\alpha(a))$. Now $GL(\alpha)$ acts on $Rep(Q, \alpha)$ so I can break it into disjoint orbits. We know that this group has only finitely many indecomposables, because their isomorphism classes are preserved by a change of basis.

Then $Dim(Rep(Q, \alpha)) = \sum_{a \in Q_1} \alpha(ta)\alpha(ha)$ and $Dim(GL(\alpha)) = \sum_{x \in Q_0} [\alpha(x)]^2$.

Now for $\alpha \in \mathbb{N}^{Q_0}$ we have $\langle \alpha, \alpha \rangle = \dim\ GL(\alpha) - \dim\ Rep(Q, \alpha)$.

The dimension of the orbit is the dimension of the group minus the dimension of the stabilizer.

Now, for $\beta \in K$ we can consider $\prod \beta I \in GL(\alpha)$. This acts trivially so we know that we have $\langle \alpha, \alpha \rangle > 0$. Then we extend to $\mathbb{Z}$ and then $\mathbb{R}$ to show positive definiteness.

If you have finitely many orbits, one of the orbits has to have dimension the same dimension as the space, since you can't cover a manifold with smaller-dimensional manifold. That's why the proof fails if you don't assume your quiver is finite type. It also gives you an estimate on the number of degrees of freedom you have in your representation.

Let's look at A_n. Here $\Phi = \{e_i - e_j\}$ for $i \neq j, 1 \leq i, j \leq n+1\}$ and $\Phi^+ = \{e_i - e_j | j > i\}$.

Then the indecomposable representations of A_n look like

$$0 \;\text{———}\; 0 \cdots\cdots 0 \;\text{———}\; k \xrightarrow{\ 1\ } k \cdots\cdots k \;\text{———}\; 0 \cdots\cdots 0$$

and this yields the correspondence.

We meet next time in two weeks, on April first.