Physics Seminar
February 25, 2005
Jaimie Thind

Gabriel C. Drummond-Cole

March 18, 2005

The references I guess are Prasolov, Sossinksi, and Kassel.

Maybe we should start by announcing the next week's talk. Next week is Gabriel Drummond-Cole, an introduction to the Khovanov invariants. This is closely related to things we will discuss today, but Khovanov's work is the next step after the Jones polynomial.

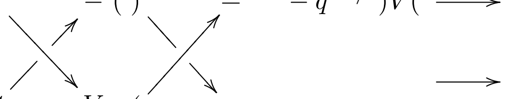
I promised examples, so we're going to discuss the Jones polynomial, and every coefficient in the Taylor expansion is a finite type invariant.

We want to assign an isotopy invariant to links which is a polynomial.

Definition 1 *The Jones polynomial invariant V is a function V : isotopy classes of links $\rightarrow$ $\mathbb{Z}[q^{-1/2}, q^{1/2}]$. If we're just talking about knots this will be in $\mathbb{Z}[q^{-1}, q]$; this will be true whenever we have an odd number of components. It satisfies*

1. $q^{-1}V(\ )$ $-(\ )$ $=^{1/2} - q^{-1/2})V(\ \ )$

   q V $($ q



2. $V(L \sqcup Unknot) = -(q^{-1/2} + q^{1/2})V(L)$
3. $V(Unknot) = 1.$

Let's look at the trefoil as an example.

[Can you motivate this?]

You start with, wanting to construct a polynomial invariant. Maybe it'll make sense by the end. You can use what's called the Kauffman bracket.

So as an example we calculate V for the trefoil. We'll expand with respect to one crossing. You get $q^{-1}V(T') - qV(T) = q^{1/2} - q^{-1/2}V(H_2)$, where T' is the unknot and H_2 is the Hopf link.

We're going to do what we did again. So for H_2, we get $q^{-1}V(H_2') - qV(H_2) = (q^{-1/2} - q^{-1/2})V(Unknot)$, where H_2' is the two component unlink. So here we can say this is $q^{-1}(q^{1/2} - q^{-1/2}) - qV(H_2) = q^{1/2} - q^{-1/2}$, which yields $V(H_2) = q^{-3/2} - q^{-1/2} + q^{-5/2} - q^{-3/2}$, i.e., $-q^{-1/2} - q^{-5/2}$. So we get $q^{-1} - qV(T) = (q^{1/2} - q^{-1/2})(-q^{-1/2} - q^{-5/2})$, so we get $V(T) = q^{-1} + q^{-3} - q^{-4}$.

This will be good enough to distinguish between the two trefoils, but it's not that great because it won't separate knots for us.

What's the connection between this and finite type knots? Look at our skein relation. If we make a substitution, $q = e^x$, then we expand e^x as a Taylor series, we get a Taylor series for V and the statement that connects with finite type invariants is, if we write $V(K)(x) = \sum V_m(K)x^m$, the statement is that $V_m(K)$ is a finite type invariant. How are we going to prove this? By looking at the skein relation. Let me remind you, we defined the extension to singular knots, and this looks like the left hand side. So the first thing to do is sub in the Taylor expansion of $e^x = q$. So we get $e^{-x}V_+ - e^xV_- = (e^{x/2} - e^{-x/2})V_0$. So we get here $(1 - x + x^2/2 + \cdots)V_+ - (1 + x + x^2/2 + \cdots)V_+ = ((1 + x/2 + (x/2)^2/2 + \cdots) - (1 - x/2 + (x/2)^2/2 + \cdots)V_0$. So the side on the right looks like $(x + \cdots)V_0$. On the other side it looks like $V_+ - V_- + x(stuff)$. So we can move this over to the other side and we get that $V_+ - V_- = x(stuff)$.

What this says is that every time we add a singular point we pick up an x, so that the mth time we do this it will be divisible by x^m. If we evaluate V on a knot with more than m singular crossings, then it is divisible by at least x^{m+1}. That means that V on a knot with more than m singular crossings has its m coefficient equal to zero.

There's an analogue of the Jones polynomial, more general, which is the Homfly polynomial. Instead of the relation, we had $q^{-1}V_+ - qV_- = (q^{1/2} - q^{-1/2})V_0$, you do the same thing but put in $-N/2$ for the -1, and $N/2$ for the 1, then all of these are also finite type invariants.

So, where are we? Scott asked why we even have a polynomial like this. I'll try to give you an idea why we should have something like this.

So draw a knot as a sequence of morphisms in a category whose objects are tensor products of V indexed by 0-dimensional subsets of the line and cobordisms in $R^2 \times I$ are morphisms. Then cups and caps interchange between k and $V \otimes V$, vertical lines are identities, and crossings are maps between $V \otimes V$ and itself. You end up with a map from k to itself. We want this to give a knot invariant. The scalar is your invariant. We need this to satisfy Reidemeister moves. We'll take $\mathbb{C}^2$ as our V, and think of it as a representation of $\mathfrak{sl}(2)$, and we want our maps to be $\mathfrak{sl}(2)$ invariant, which gives us restrictions.

So if V is $\mathbb{C}^2$ we will take $\cup(1) = e_1 \otimes e_2 - e_2 \otimes e_1, \cap$ as the unique bilinear $\mathfrak{sl}(2)$ invariant thing. This gives $\cap(e_2 \otimes e_1) = 1$. We'll take $R(x \otimes y) = y \otimes x$.

So does this actually give a knot invariant? One thing to check is that it satisfies the Reidemeister moves. For Reidemeister 0 we get $e_i \to e_i \otimes e_1 \otimes e_2 - e_i \otimes e_2 \otimes e_1 \to \delta_{i,1} e_1 + \delta_{i,2} e_2$.

For time reasons, you can check that it satisfies the other Reidemeister moves. The second one is easy, it's an invertibility argument. The third one comes from the fact that R satisfies the Yang Baxter equation. So what invariant is this going to give? When you change your crossings, the value of the invariant is not going to be affected.

So we want $R : V \otimes V \to V \otimes V$ with $R^2 \neq id$ and $(R \otimes id)(id \otimes R)(R \otimes id) = (id \otimes R)(R \otimes id)(id \otimes R)$. So we slightly change what we have here for our maps. "Deform" R by q. It may be sloppy terminology. We want $\cup(1) = e_1 \otimes e_2 - q^{-1} e_2 \otimes e_1$. We will have $\cap(e_1 \otimes e_2) = -q$ and $\cap(e_2 \otimes e_1) = 1$. The intereseting part is that R, as a matrix with respect to $e_i \otimes e_j$ with the dictionary ordering, is

$$\begin{pmatrix} q^{1/2} & & & \\ & q^{1/2} & & \\ & & q^{-1/2} & q^{-1/2}(q - q^{-1}) \\ & & & q^{-1/2} \end{pmatrix}$$

times the original flip. You can check the Reidemeister moves. The third one comes from the Yang Baxter equations. At this point we don't have a knot invariant because it doesn't satisfy the first Reidemeister move. It depends on framing.

There is a full solution to the Yang Baxter equation in some small degree, but this is really the wrong question, it is like asking whether there is a classification of the Jacobi identity solutions to classify Lie algebras.

This associates to a knot an invariant. The output is a Laurent polynomial in $q^{1/2}$. The ground ring is Laurent polynomials. We can check that this satisfies the defining relations. We can split $V \otimes V$ into $\mathbb{C} \oplus \mathbb{C}^3$. So the three entries in the skein relation have to have a linear relation since $\mathfrak{sl}(2)$ invariant maps from $V \otimes V$ to itself are two dimensional. That is exactly the relation for the Jones polynomial after suitable rescaling.

Another thing to note is that for $\mathfrak{sl}(n)$ this gives the Homfly. The magical q stuff comes from looking at the quantum group $U_q\mathfrak{sl}(2)$. These are invariant maps for this quantum groups.