Physics Seminar
February 18, 2005
Jaimie Thind

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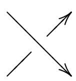
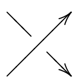
This is Jaimie Thind, he's going to talk about knots and finite type invariants.

So does everybody know what a knot is? Yes? No?

Definition 1 *For me a knot will be a smooth embedded S^1, so a circle smoothly embedded in $\mathbb{R}^3$.*

Sometimes you let it be piecewise linear but for me everything will be smooth.

A knot diagram is a projection of your knot onto a plane, making sure it's in general position so that you don't introduce singularities. When there's crossings, you have something that looks like  or  . So the intersection point comes from two different points in the z coordinate.

So you draw everything in the plane because it's easier to work in the plane.

Definition 2 *Two knots K_0 and K_1 are equivalent if there is a smooth isotopy of $f_t : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f_1(K_0) = K_1$ and $f_0 = id$. Here f_t is a diffeomorphism for all t.*

You want to study these knot diagrams, and there are these things called Reidemeister moves. These are local changes to knot diagrams.

[Pictures of the Reidemeister moves]

Reidemeister showed that two diagrams are equivalent if and only if they differ by a finite number of these and an isotopy of the plane.

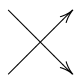
Theorem 1 *(Reidemeister) Two diagrams correspond to equivalent knots if and only if one can be brought to the other by a finite number of Reidemeister moves and a plane isotopy.*

I'm not going to prove this, it's in any book on knots.

The next thing I want to introduce is the idea of a singular knot.

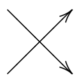
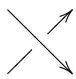
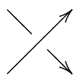
[You might say, this is something like Morse theory; you can build up the isotopy out of these at the most degenerate, you don't need more than three strands.]

Definition 3 *An m singular knot is a smooth immersion of S^1 in $\mathbb{R}^3$. with m "singular" points, i.e., transversal self-intersections.*

So locally they can look like  .

These knots come with an orientation from the positive orientation of S^1. Now we get to the point of the talk, which is to talk about finite type invariants.

If we just start with a, if V is numerical, say, $\mathbb{R}$-valued invariant, then we can extend it to singular knots by the following rule:

$V^{(1)}($  $) = V($  $) - V($  $)$

This is for one singular point. For m singular points you just repeat.

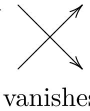
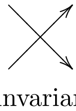
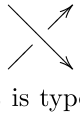
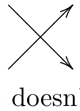
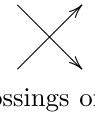
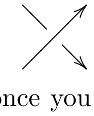
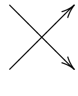
You kind of want to say that these are like derivatives.

[You also need to check that these are well-defined, that it doesn't matter which way you resolve.]

Definition 4 *V is a type m invariant if $V^{(m+1)}$ is identically zero. It vanishes on anything with $m+1$ singular points.*

Let V_m be the space of type m invariants; then $V_0 \subset V_1 \subset \cdots$

Type zero invariants are constants. They're sort of trivial. So you can use the same idea, if you have a knot with m singular points, it doesn't see how the rest of the knot looks in some sense.

So $V^{(m)}($  $\ldots^m\ldots$   $) - V^{(m)}($  $\ldots^m\ldots$   $) = V^{(m+1)}($  $\ldots^{m+1}\ldots$

but this vanishes if your invariant is type m. So it doesn't see crossings once you get to m singular points.

So a type m invariant on an m-singular knot really only sees how the singular points lie on your S^1 before the projection. So to simplify things we're going to look at chord diagrams.

Definition 5 *A chord diagram (CD) is an oriented S^1 with finitely many chords marked on it (all of the endpoints are distinct), considered up to orientation-preserving diffeomorphism (of S^1).*

There's a way to pair these with m-singular knots up to crossing changes.

Let $\mathscr{D}^c$ be the set of nonequivalent chord diagrams, it is graded by the number of chords. So $\mathscr{G}_m\mathscr{D}^c$ is diagrams with m chords.

So $\mathscr{G}_3\mathscr{D}^c$ has a diagram with three nonintersecting "parallel" chords, a triangle of nonintersecting chords, an isolated chord and a pair, two that don't intersect and one that intersects both, or three all of which intersect one another.

So now you think of this circle as parameterizing your knot. Let's draw a couple of pictures here. You can go the other way; if you start with a knot, you can get a chord diagram, putting in a chord for every singular point.

Now we want to change the language over and just talk about chord diagrams. So corresponding to invariants we will get weight systems.

Definition 6 *An F-valued weight system of degree m is a function $W : \mathscr{G}_m\mathscr{D}^c \to F$ satisfying two rules involving pictures:*

1. *the 4T relation*
2. *the 1T, W vanish on a chord diagram with an isolated chord.*

These relations come from relations that these finite type knot invariants satisfy, which is, let's see, I'm going to try to draw, and I'm sorry, I'm terrible.

So the 4T corresponds to the third Reidemeister move and the 1T corresponds to the first.

The next thing you do is look at the vector space spanned by these chord diagrams, and a weight system will be a functional on that vector soace.

[So maybe a precise statement.]

So $\mathscr{A}^c = \langle \mathscr{D}^c \rangle / 4T, 1T$. This is a graded vector space with an algebraic structure. The point is the theorem

Theorem 2 *Let $F = \mathbb{R}$.*

1. *Given m there exists a naturally defined map $V_m \ni V \to W_m(V) \in (\mathscr{A}^c_m)^*$ which associates, well, a type m invariant to something in the dual of the mth graded piece.*

2. *There is a naturally defined map* $(A_m)^* \ni W \to V(W) \in V_m$.

3. *These are almost inverse; what I mean is that* $W = W_m(V(W))$ *and* $V = V(W_m(V))$ *up to a type* $m-1$ *invariant.*

This is like if you're doing sort of differentiation and integration. If you differentiate and then integrate, the difference is up to a polynomial of order one less than the number of times you've integrated.

[An immediate corollary is that the quotient by the $m-1$ level invariants is naturally isomorphic.]

So if you start with an invariant and you want to come up with a functional or one of these weight systems, what is the map $V \to W_m(V)$? So $W_m(V)(D) = V(K_D)$ where K is any knot corresponding to the diagram D. This is well-defined because V doesn't see any other knotting. The kernel ker W_m is things that vanish on m-singular knots, which is exactly V_{m-1}.

Next week will be just examples, so maybe this is a good time to stop.

[Let me add some comments: the point is that these 4T and 1T relations are the only ones, anything which satisfies these corresponds to this kind of invariant. This second direction is extremely difficult. So you need to construct a finite type invariant, and there are several constructions, the most famous of which is the Kontsevich integral, which is hard.

There are only finitely many in each degree because you only have finitely many chord diagrams, so these are easier to deal with than all functions, and they naturally appear in many cases. Many invariants are finite type like this. A natural question is whether they distinguish knots. The answer is unknown. That is the main open question.]

Dror Bar-Natan "On the Vassiliev Invariants," "finite type Invariants"
Prasolov and Sossinski "Knots, links,..."