Physics Seminar
February 11, 2005
Leon Takhtajan

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Our first speaker is Leon Takhtajan.

There is no official title, this is an introduction to Lagrangians and action functionals, well, the action principle. I will try to explain what this is and motivate, but first we need terminology.

We'll be dealing either with classical mechanics or classical field theory. In classical mechanics, we have an ambient space M, a finite dimensional manifold called configuration space to model the places a particle can be. So then you consider all maps $I \to M$, where you think of I as time. So the main object is $P(M)$, the path space. They call this $0+1$ theory, where the source is one dimensional.

In classical field theory you consider maps from $X \times \mathbb{R} \to M$, where X is d-dimensional. So this is $d+1$ theory.

There is a big difference between a particle and a field, because a field lives in all points in space for all time. So we consider $Map(X \times \mathbb{R}, M)$ or $Map(\mathscr{X}, M)$, where this is called space-time. So now this can be a general $d+1$-dimensional manifold. Now consider a vector bundle $\begin{array}{c} E \\ \big\downarrow \\ X \end{array}$ over the space and consider sections $\Gamma(\mathscr{X}, E)$. So we want to replace $X \times \mathbb{R}$ with the sections. When $E = \mathscr{X} \times \mathbb{R}$ you get functions on space-time. Or you replace $\mathbb{R}$ with $\mathbb{R}^n$ you get vector valued functions.

So the Lagrangian will give us the best understanding of the dynamics of these, classical mechanics and classical field theory.

For classical mechanics, if you know the location and velocity of all your particles, then you know the locations for all time. This is surprising, why don't you need to know accelerations, why do you need both of these. You characterize your motion by purely mathematical principle, called the action principle or the least action, which you can describe as follows.

You have M and the tangent bundle TM, whose elements are (q, v). Now we want a Lagrangian $L : TM \to \mathbb{R}$. Consider any path $\gamma \in P(M)$. These are all parameterized paths over $[t_0, t_1]$. So you have a path $\gamma(t) \in M$. Then the action $S : P(M) \to \mathbb{R}$ corresponding to the Lagrangian L is as follows. At every point t there is a tangent vector $\dot{\gamma}(t) \in T_\gamma(t)M$, so then we define$S(\gamma) = \int_{\gamma(0)}^{\gamma(1)} L(\gamma(t), \dot{\gamma}(t))dt$.

Then the principle of the least action goes back to Lagrange, Laplace, other people, it's very old. It says that critical points, let me say it better. The actual trajectory from q, I forgot a very important thing. The end points of the interval were supposed to be mapped to marked points on the manifold. So we have $\gamma(t_i) = q_i$, and the action principle says the actual trajectory which starts at q_0 at time t_0 and ends at q_1 at t_1 is the critical point of S. This is a bit misleading because the critical point may not be a minimum, it should be extremal action.

From $S(\gamma)$ we can solve to see what is the critical point, this is called the calculus of variations. You get a second order ODE, and now we can ask, why not first or third order? Say you want to generalize.

Exercise 1 *Derive this system of differential equations, called Euler-Lagrange equations.*

Exercise 2 *Consider the following:* $S(\gamma) = \int_{t_0}^{t_1} L(\gamma, \dot{\gamma}, \ddot{\gamma}, \cdots, \gamma^{(M)})dt$*, then you get an ODE of order* $2M$.

This was explained by Laplace and others; he said that our world was the best because the principle of the least action holds.

I will give you a side remark, what do you do with differential equations of odd order? These can be obtained from the action principle if one uses, instead of normal variables, anticommuting variables or Grassman variables. Then you can get, that explains why we don't see that around us.

[Can you really say what is action in some intuitive way? We all know what is energy.]

Energy is kind of secondary, you say that energy is conserved along the motion but this is a secondary notion to the laws governing the motion. We derive how the particle moves which indicates how to find the energy. If you go along each choice you expend a certain amount of effort. Here you choose the easiest way, you don't do it in life but here it is exactly what you look for.

[Is the Lagrangian the force?]

You get force and also the difference in energy. Lagrangian is more fundamental, because it includes the energy. You can do a Hamiltonian consideration, where you start from energy. We want to derive differential equations to solve later. We need to write down equation explicitly. If you want to put your theory on a foundation, you have to start somewhere; this is the most basic place to start and you take this on faith from experimentalists.

Exercise 3 *Let $M = \mathbb{R}^3$; then let $L(q,v) = \frac{m||v||^2}{2} - U(q)$. Here $||v||^2 = \sum v_i^2$; now the first term is kinetic and the second is potential. Derive $m\vec{a} = \vec{F} = -\frac{\delta U}{\delta \vec{q}}$.*

It turns out there are special examples to describe what happens experimentally.

Action is kind of a canonical thing, a function; then there are critical values of this function which are different things. In Riemannian geometry you consider a point and then all possible,

Exercise 4 *Another interesting example, in a Riemannian manifold, look at M and take $L = ||v||^2$, and derive geodesics.*

Draw all possible arrows out of the origin. Then $S(q,t)$ is related to a symplecomorphisms. This gives you a transformation from the initial to final values along a symplectomorphism preserving the Hamiltonian.

Okay, now let's describe briefly the field theories. So you see classical mechanics studies ODEs. This gives you local information. If you want global considerations, you need to look at global properties. This approach knows nothing about topological invariants. But when you look at classical field theory, you can probe your space time and find out things about it, by considering different field theories on this source space. In classical mechanics the source is the real line. In many cases you study what is induced on the source.

Examples of classical field theories, where we are dealing with PDEs, since they vary with t and with coordinates.

Let $\mathscr{X}$ be $\mathbb{R}^4$ and $E = \mathbb{R}^4 \otimes \mathbb{R}$ the trivial line bundle. Let sections be fields and then the simplest example are the Klein-Gordon equations, which are in this form:

$$g^{ab}\frac{\delta\phi}{\delta x^a}\frac{\delta\phi}{\delta x^b}m^2\phi$$

g_{ab} is a psuedo-Riemannian metric and g^{ab} the inverse of this matrix.

So take $g^{ab} = \delta ab - Euclid$, so $g^{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$. We can get elliptic, Euclidean, $\Delta\phi + m^2\phi = 0$, or Minkowski, where we have $\Box\phi + m^2\phi = 0$. This is the hyperbolic setting. Poincaré said to look at the symmetry group of something like this, inventing what Einstein did independently.

You can describe four dimensional hyperbolic PDEs as three dimensional elliptic PDEs.

You can formalize some of this, but it's a mess because you need to work in infinite dimensional manifolds. Look to Quantum Fields and Strings, v.1., Deligne and Freed.

Now L the Lagrangian depends on $\phi(x)$ and $\delta_a\phi(x)$ and is $\frac{1}{2}g^{ab}\delta_a\phi\delta_b\phi - \frac{m^2\phi^2}{2}$.

Now $S = \int_{\mathbb{R}^4} L(\phi, \delta_a\phi)d^4x$.

Exercise 5 *Do this calculus 125 problem you have to know integration by parts, well, Stokes' theorem.*

Field theory is much richer than classical mechanics. You can get linear or nonlinear differential equations.

Intuitively your field is infinitely many particles; the m is the mass of this many particles.

Consider the simplest example of this type, $\phi_{xx} - \phi_{tt} = 0$, thhen you get $\phi(x,t) = f(x+t) + g(x-t)$. If you add m you get decay. In general it's not easy to say what the Lagrangians are, or whether they exist, for certain PDEs. Let's discuss instead that you can move in two dimensions. You can consider nonlinear equations or go in the opposite direction to look at PDEs that are simple and have a topological interpretation. One example is the Navier Stokes equation.

Let's discuss, briefly, the Yang Mills equation and characteristic classes. I want to describe the Wen Zumero Welter action functional. This is easier because things are geometric. You have to do the same calculations, which I will skip.

There's another example, most difficult, consider a Riemannian manifold (M, g), all possible metrics, then $S(g) = \int_M R\sqrt{g}$, the Hilber Einstein action

Say you have a bundle of E over M and a connection in the bector bundle E, then you define the F-curvature of A to be $F_A = \delta A + [A, A]$, a 2-form.

Now define $c_i(A) = tr(F \wedge \cdots \wedge F) \in \Omega^{2i}(M)$. Then take $Y \in H_{21}(M)$ and look at $\delta \int_Y c_i(A) = 0$.

If M is four dimensional, look to $c_2 = \int_M tr\ F^2$; this is a topological iinvariant.

If E is a line bundle then $[,]$ is a bracket in the corresponding Lie algebra and you get 0, these are Maxwell's equations.

Now let's change it slightly. Suppose M is Riemannian. Then a Riemannian metric introduces a Hodge star operator on A. So $c_2(A) = \int_M tr(F_A \wedge F_A)$ and $YM(A) = \int_M tr(F_A \wedge *F_A)$. Bow $\delta YM(A) = 0$ is the Yang Muls equations.

Exercise 6 *Define the Yang Mills equations.*

For the special case, look to the self-dual Yang Mills equations. The global minimum consists of the Yang Mills equations.

The last example is the Wen Zumeno Welten function. A very good point is that when E is one-dimensional this is precisely–the Yang Mills equation reduced to Maxwell.

Let's look at this last example. These are called chiral fields. By definition we have $g : S^2 \to G$ a simple Lie group. Let S^2 be the compactification of $\mathbb{R}^2$. But in two dimensions you get something interesting, dgg^{-1} the Mauerer Cartan form of G. Then you define the functional explicitly as follows. $S(g) = \int_{S^2} ||dg\ g^{-1}||^2$. Say ω is a Maurer-Cartan 1-form on G with values in a Lie algebra. Then $dgg^{-1} = g^*(\omega)$, the pullback. So ω is the unique left-invariant one form so that at the identity $\omega_e(X) = X$. It's easy as that.

Okay, so that's the function.

Exercise 7 *I'll give as an exercise to derive corresponding Euler Lagrange equations.* $\delta S = 0$ *leads to, let* $A = A_0 dx^0 + A_1 dx^1$. *Then it is* $\frac{\delta A}{\delta x_0} + \frac{\delta A}{\delta x_1} = 0$ *and* $\frac{\delta A_1}{\delta x_0} - \delta A_0 \delta x^1 + [A_1, A_0] = 0$.

Exercise 8 *Show that the connection is flat, from the Maurer Cartan equation.*

The norm is an invariant Euclidean inner product given by the Killing form. Then I can write $\langle X, Y\rangle = tr(XY)$. When it is simple this is positive definite.

I need a metric on the sphere. We want to, our equation looks like $\frac{\delta}{\delta x^0}(\frac{\delta g}{\delta x^0} g^{-1}) + \frac{\delta}{\delta x^1}(\frac{\delta g}{\delta x^1} g^{-1})$ and let's say you want to put these in terms of $z = x^0 + ix^1$, you can get $\frac{\delta}{\delta \bar{z}}(\frac{\delta g}{\delta z} g^{-1}) = 0$.

Let's define another function calld the Wen Zumiro functional. We define $W(g) = \int_{B_\gamma} \Omega$, since G is simple, we have $H^3(G, \mathbb{R}) = \mathbb{R}$. There's a canonical Ω, a left invariant three form, and $\Omega_e(X, Y, Z) = \langle [X, Y], Z\rangle$ and this is unique. Then $d\Omega = 0$ and $[\Omega]$ generates $H^3(G, \mathbb{Z})$. Consider $\gamma = g(S^2)$, and since $\pi_2(G) = 0$, there exists a three-chain B_γ with γ as its boundary. Then the fundamental fact is that if you replace B_γ with another, then the difference of the correspondiong functions will be of an integer. This is not a single-valued function but its variation is simple.

So the function, add it $S(g) + \alpha W(g)$, the WZW functional. I'll leave everything as an exercise, let me write it down nicely. What do you get? The exercise will be, prove that the variation of this function is zero. This implies that $[A_1, A_0] = 0$. We need the definition of Ω. Using this, if you consider the 2-terms, one of which comes from energy and the other of which is topological, then you can get equations of motion, the first one won't change but the second one eliminates the commutator. So you get the equations of motion, related to conformal field theory, Kac-Moody Lie algebras, and so on. There are other examples in two dimensions, and if you want to study extra topological properties one should look at Grassman variables. This uses data on the four-manifold so that this functional does not depend on the metric. In two dimensions the things are really nice and that's a good topic, to study this model in detail. I'd better stop, thank you.