Physics Seminar
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We will begin then. Let me start by reminding you what we did last time. Last time I discussed $U_q\mathfrak{sl}_2$, the q-deformation of $\mathfrak{sl}_2$, whatever this is, it is something where you have this relation $[e,f] = \frac{q^h - q^{-h}}{q - q^{-1}},\ [h,e] = 2e \iff q^h e = q^2 e q^h,\ [h,f] = -2f \iff \cdots$, and we have representations V_n, and a representation with highest weight n is generated by v_0 up through v_n, with the action of e and f given by $ev_i = [i+1]v_{i+1}$ and $fv_j = [n-j+1]v_{j+1}$ with $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

So $V_1 = \mathbb{C}^2$, with $ev_- = v_+,\ fv_+ = v_-,\ hv_\pm = \pm v_\pm$.

We also defined the tensor product of representations. The action of the tensor product is given by the formulae
$\Delta e = e \otimes q^h + 1 \otimes e$,
$\Delta f = f \otimes 1 + q^{-h} \otimes f$,
$\Delta h = h \otimes 1 + 1 \otimes h$.

We discussed how to do $V_1 \otimes V_1$. We discussed in general how $V_n \otimes V_m = \oplus c_{nm}^k V_k$, and the c_{nm} are not dependent on q. To see the embedding explicitly, on the other hand, you'll get a lot of coefficients. They are close to the usual $\mathfrak{sl}_2$ but your binomial coefficients $\binom{a}{b}$ should be replaced with the q-analogue defined as $\frac{[a]!}{[b]![a-b]!}$, where $[n]! = \prod_{i=1}^n [i]$. You also have to put in appropriate places various powers of q.

At this moment you might think the story reduces to taking the usual representation theory and replacing usual numbers with q-numbers.

The big difference is that the factors are not symmetric. You might think that by a minor change of variables you could get rid of this, but you can't.

So take $V = \mathbb{C}^2$, and consider V^*. I said last time you can define a structure of a representation on it, by requiring that $V^* \otimes V \to \mathbb{C}$ be a morphism of representations. You want $\Delta(e)(v^* \otimes v) = 0$, since e acts by 0, and similarly for f and h. If you write these explicitly it gives the action in the dual space. If you write it in the dual basis for the basis given before, you get $hv_+^* = -v_+^*,\ hv_-^* = v_-^*$, so you might expect that everything will just be reversed. You get $ev_+^* = -qv_-^*,\ fv_-^* = -q^{-1}v_+^*$.

There is also the canonical map $\mathbb{C} \to V \otimes V^*$, which is also a map of $U_q\mathfrak{sl}_2$-modules. You have to be careful about the order.

Now suppose that I took the same dual space, and instead of requiring that these commute with the action, instead I required that $V \otimes {}^*V \to \mathbb{C}$ commute with the action. Here ${}^*V$ is V^* as vector spaces. For a classical Lie algebra permutation of factors does not preserve the action.

Again I will skip ahead. For h nothing changes, which is to be expected, but $ev_+^* = -q^{-1}v_-^*$ and $fv_-^* = -qv_+^*$. I don't want to do the computations so let me leave it as an easy exercise. If I use these formulas, then the map $\mathbb{C} \to {}^*V \otimes V$ is a map of $U_q\mathfrak{sl}_2$-modules. So we can define the action on the dual space in two ways. The actions do not coincide. We have the left dual and the right dual. But they are isomorphic, just not as naively as one might hope. Instead take v_+^* to $q^{-1}v_+^*$ and $v_-^* \to qv_-^*$. For example you need to check the commutativity of the square

$$\begin{tikzcd} v_+^* \arrow[r] \arrow[d, "e"'] & q^{-1}v_+^* \arrow[d, "e"] \\ -q^{-1}v_-^* \arrow[r] & -v_-^* \end{tikzcd}$$

You can write this uniformly as $v \to q^h v$.

As a corollary, let's talk about quantum dimension.

One way to define dimension of a vector space is that $\dim\ V = tr\ id_V$. So the identity operator, if this is seen in $V \otimes V^*$, is the image of the unit $\mathbb{C} \to V \otimes V^*$, and the trace is the map in the other direction.

In representations of $U_q\mathfrak{sl}_2$ I can do $\mathbb{C} \to V \otimes V^*$ but I need to get from there to $V \otimes {}^*V$ in order to get back to $\mathbb{C}$. So we move via $1 \otimes q^{-h}$. So this takes $1 \to v_+ \otimes v_+^* + v_- \otimes v_-^*$, which goes to $v_+ \otimes qv_+^* + v_- \otimes q^{-1}v_-^*$, which goes to $q + q^{-1} = [2]$.

For any irreducible, which has dimension $n+1$, the q-analogue has dimension $[n+1]$.

Now let me talk about the most famous corollary of this. Start now with any two representations V and W. I can consider $V \otimes W$ and also $W \otimes V$. As vector spaces there is a canonical isomorphism between them, where you just permute the factors. Normally people wouldn't distinguish between them. If I consider, however, each as representations of $U_q\mathfrak{sl}_2$, then this permutation is not a morphism.

This is because $P(e(v \otimes w)) \neq eP(v \otimes w)$. But these are $q^h w \otimes ev + ew \otimes v$ and $ew \otimes q^h v + w \otimes ev$.

These two expressions don't match, so this is not a morphism. So these are not obviously isomorphic. On the other hand, if I take these to be irreducible, then each breaks up into irreducibles with the appropriate dimensions, so they are isomorphic. Is there a way of constructing a natural isomorphism? The answer is yes, and this something is the famous R-matrix. Let me write a theorem.

Theorem 1 *Drinfeld*
There is a unique element $R = q^{\frac{h\otimes h}{2}}(q + c_1 e\otimes f + \ldots + c_n e^n \otimes f^n + \ldots)$ in the appropriate completion of $U_q\mathfrak{sl}_2 \otimes U_q\mathfrak{sl}_2$ (well-defined in any finite dimensional representation without the completion) such that $R^{\checkmark}_{V,W} = PR : V\otimes W \to W \otimes V$ is a morphism of $U_q\mathfrak{sl}_2$-modules.

Namely, $c_n = \frac{q^{\frac{n(n-1)}{2}}(q-q^{-1})^n}{[n]!}$.

How do you prove this theorem? Once you know what you're looking for, it's not very hard. Drinfeld's proof uses the quantum double, which is too hard, but if you know what you're looking for, well, you write things like

$$P(e\otimes q^h + 1\otimes e)PR = R(-e\otimes q^h + 1\otimes e),$$

or

$$(q^h \otimes e + e\otimes 1)R = R(e\otimes q^h + 1\otimes e).$$

You get some relations on c_1, and do some calculations, and this is what you get.

As an example, for $V = W = V_1$, then we get $R = q^{\frac{h\otimes h}{2}}(1 + (q-q^{-1})e\otimes f)$. The terms after these involve e^2 and f^2, which act by zero. Then $R^{\checkmark}$ acts as the following: $v_+\otimes v_+ \mapsto$
$q^{1/2}v_+\otimes v_+$
$v_+\otimes v_- \mapsto q^{-1/2}v_-\otimes v_+$
$v_-\otimes v_+ \mapsto q^{-1/2}v_+\otimes v_- + q^{-1/2}(q-q^{-1})v_-\otimes v_+$
$v_-\otimes v_- \mapsto q^{1/2}v_-\otimes v_-$

In the limit where q goes to 1, the c_n go to 0 because of the $q - q^{-1}$, so the $R^{\checkmark}$ goes to the permutation matrix. This has the form, as a four by four matrix,

$$q^{1/2}\begin{pmatrix} 1 & & & \\ & 0 & q^{-1} & \\ & q^{-1} & q^{-1}(q-q^{-1}) & \\ & & & 1 \end{pmatrix}$$

One thing to see is that $(R^{\checkmark})^2$ is not the identity. Recall that $V_1\otimes V_1$ splits into $\mathbb{C}\oplus V_2$, this last being spanned by $v_+\otimes v_+, v_-\otimes v_-$, and $v_-\otimes v_+ + q^{-1}v_+\otimes v_-$. The one dimensional space is spanned by $v_+\otimes v_- - q^{-1}v_-\otimes v_+$. So $(R^{\checkmark})^2$ gives constants on each of these spaces. If you believe it acts by a constant, then it acts by q on the V_2 piece. For $\mathbb{C}$,

$$R^{\checkmark}(v_+\otimes v_- - q^{-1}v_-\otimes v_+) = q^{-1/2}v_-\otimes v_+ - q^{-1}(q^{-1/2}v_+\otimes v_- + q^{-1/2}(q-q^{-1})v_-\otimes v_+))$$
$$= -q^{-3/2}v + \otimes v_- + (q^{-1/2} - q^{-1/2} + q^{-5/2})v_-\otimes v_+$$

$$= -q^{-3/2}(v_+ \otimes v_- - q^{-1} v_- \otimes v_+).$$

The eigenvalue for $(R^{\checkmark})^2$ has eigenvalue q^{-3}.

Again you have a rather rich structure. The tensor product is weakly symmetric, since the isomorphism is nontrivial. Now let me answer Gabriel's question, why I don't kill $q^{1/2}$ by multiplying by the appropriate power. Suppose I have the tensor product of three representations, and I want to interchange V_1 with the tensor product of V_2 and V_3.

$$\begin{tikzcd}
V_1 \otimes V_2 \otimes V_3 \arrow[r] \arrow[d] & (V_2 \otimes V_3) \otimes V_1 \\
V_2 \otimes V_1 \otimes V_3 \arrow[ur] &
\end{tikzcd}$$

This diagram is usually called the hexagon equation because I am implicitly using the associativity of the tensor product. This is a rather natural thing to require, and for quantum groups as deformations like this, that this holds but I don't want to multiply by a q factor.

There could be a different R matrix as long as it did not have the same form in the completion. I would rather focus on this question later. If you do it properly as a power series over $\log\ q$, there is only one, but that's a much deeper result.

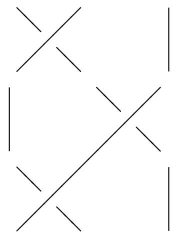
As a corollary, you get the Yang Baxter equation. I will do the diagram.

$$\begin{tikzcd}
 & V_1 \otimes V_3 \otimes V_2 \arrow[r] & V_3 \otimes V_1 \otimes V_2 \arrow[dr] & \\
V_1 \otimes V_2 \otimes V_3 \arrow[ur] \arrow[dr] & & & V_3 \otimes V_2 \otimes V_1 \\
 & V_2 \otimes V_1 \otimes V_3 \arrow[r] & V_2 \otimes V_3 \otimes V_1 \arrow[ur] &
\end{tikzcd}$$

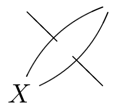
The theorem is that these commute.

I said that this is a corollary. Let me show how you get from the theorem to the corollary, in the language of diagrams. For the R-matrix, you have this property. If you have an operator from V to V which you follow by the R-matrix, then you can move it past, i.e., $R^{\checkmark}(\phi \otimes id) = (id \otimes \phi)R^{\checkmark}$. Why is this? Since ϕ commutes with e, f, h, which make up R, it commutes with R. Then it moves through P in the desired way, as everything does.

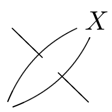
In abstract nonsense, $R^{\checkmark}$ is a functorial isomorphism.



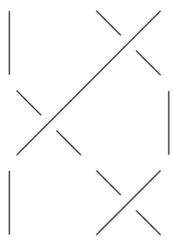
Is the left hand side, and I can bunch together to get



Then this is



which is



Written $R_i^{\checkmark} R_{i+1}^{\checkmark} R_i^{\checkmark} = R_{i+1}^{\checkmark} R_i^{\checkmark} R_{i+1}^{\checkmark}$, this is the braid relation. This defines a representation of the braid group. From there it is relatively straightforward to get invariants of knots.

Let me explain one last thing. It is good that we don't have $(R^{\checkmark})^2 = 1$ from this point of view; if you had this equal to one, you could unlink your braids and you would only remember the end points.