Physics Seminar
April 22, 2005
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April 27, 2005

Roughly, a quantum group is a "q-deformation" of a usual classical group. You want the specialization $q = 1$ to be the usual group, but how to do that is obviously a harder question. This topic was maybe a little too hot in the 1990s and now is not so hot any more. A good book to start with is Jantzen, "Lectures on Quantum Groups," and Kassel, "Quantum Groups." This book starts in the wrong place, the first example appears on page two hundred and something.

What are quantum groups? They are q-deformations of usual groups. Unfortunately, you can't do a normal deformation. The group $GL(n)$ cannot be deformed as a group. So quantum groups are not groups. This is a rather loose term, there are many types of quantum groups, but the most common type is quantized universal enveloping algebras $U_q\mathfrak{g}$, where $\mathfrak{g}$ is a semisimple Lie algebra.

Today we will be talking about an even simpler case, where $\mathfrak{g}$ is $\mathfrak{sl}(2)$. So $U_q\mathfrak{sl}(2)$ is not a group or a Lie algebra, but it has a nice representation theory. That's the important part, whatever words you need to put in front to make this a nice representation theory, you do this.

Recall that $\mathfrak{sl}(2)$ is traceless 2×2 matrices, everything is over the complexes today, generated by e, f, and h subject to the relations $[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f$.

If ∂_i denotes $\frac{\partial}{\partial x_i}$. So $e = x_1\partial_2,\ f = x_2\partial_1,\ h = x_1\partial_1 - x_2\partial_2$. For the obvious action of $GL(2)$ on $\mathbb{C}^2$ you get this action of the Lie algebra by vector fields.

If you consider the action of this Lie algebra on the space V_n of polynomials of degree n in $x_1,\ x_2$, in particular you can compute what this action looks like in a basis. So $e(x_1^n x_2^m) = mx_1^{n+1}x_2^{m-1}$. This is, I believe, covered in MAT 123. $f(x_1^n x_2^m) = nx_1^{n-1}x_2^{m+1}$ and $h(x_1^n x_2^m) = (n - m)x_1^n x_2^m$.

We can make a picture:

$$x_2^n \qquad \cdots \qquad x_1^{n-2}x_2^2 \underset{2}{\overset{n-1}{\rightleftarrows}} x_1^{n-1}x_2 \underset{1}{\overset{n}{\rightleftarrows}} x_1^n$$

where the upper arrows are the action of f, the lower arrows the action of e.

So in the quantized sense we have $\partial_t \rightsquigarrow D_t = \frac{f(t+h)-f(t-h)}{2h}$. This leads, via $x = e^t$ and quantization, to $D^q = \frac{f(qx)-f(q^{-1}x)}{(q-q^{-1})}$.

Let's replace derivatives with differences and denote the result $U_q\mathfrak{sl}_2$. We let $e = x_1 D_2,\ f = x_2 D_1$. h we will leave blank; the trick here is to know what to deform and what to keep the same.

$[e, f] = \frac{T_1 T_2^{-1} - T_2 T_1^{-1}}{q-q^{-1}}$.

$$Dx = \frac{(xf)(qx) - (xf)(q^{-1}x)}{(q-q^{-1})x} = \frac{qf(qx) - q^{-1}f(q^{-1}x)}{q-q^{-1}} = \frac{qT - q^{-1}T^{-1}}{q-q^{-1}}.$$

$$xD = \frac{f(qx) - f(q^{-1}x)}{q-q^{-1}} = \frac{T - T^{-1}}{q-q^{-1}}.$$

Now if $h = x_1\partial_1 - x_2\partial_2$ then we get $[e, f] = \frac{q^h - q^{-h}}{q-q^{-1}}$.

So how does this act on the space of polynomials? $D^q(x^n) = \frac{(qx)^n - (q^{-1}x)^n}{(q-q^{-1})x} = \frac{q^n - q^{-n}}{q-q^{-1}} x^{n-1}$. This fraction is often called the q-analogue of the number n and is denoted $[n]$. Its expansion is $q^{n-1} + q^{n-3} + \ldots + q^{-n+1}$. When $q = 1$ this is n.

So $ex_1^n x_2^m = [m]x_1^{n+1}x_2^{m-1},\ fx_1^n x_2^m = [n]x_1^{n-1}x_2^{m+1},\ hx_1^n x_2^m = (n-m)x_1^n x_2^m,\ q^h(x_1^n x_2^m) = q^{n-m}x_1^n x_2^m$.

I should write the commutation relations for the other combinations; it turns out that these are unchanged: $[h, e] = 2e,\ [h, f] = -2f$. This is equivalent to $q^h e = q^2 e q^h$.

[Why don't you deform the h?]

The short answer is that then you don't get anything interesting. The long answer is that these three vectors are not all the same. It turns out you need to fix a Cartan subalgebra for this to make sense, and here that is $\langle h \rangle$.

I can formulate a theorem now that I don't want to prove:

Theorem 1 1. *For $\mathfrak{sl}_2$, all irreducible finite dimensional representations are of the form V_n, constructed above. This is a known classical result, going back to I don't know whom.*

2. *The same holds for $U_q\mathfrak{sl}_2$, if I consider the associative algebra generated by e, f, h, assuming h is diagonalizable, (otherwise q^h makes no sense).*

This works similarly to the classical case, by finding a highest weight and so on. At the induction step, or some moment you need to use $[m][n+1]-[m+1][n]$. These are not $[m(n+1)]$ or $[(m+1)n]$. They are not remotely equal by degree conditions. But this difference is $[m-n]$.

This was, so far I haven't done much. I've defined representations. But now let's move to the tensor product. There $\partial(fg) = (\partial f)g + f\partial g$ so $e.fg = (ef)g + f(eg)$. Since you might realize that f and g are placeholders, you get the same thing for the tensor product. $\Delta e = e \otimes 1 + 1 \otimes e$. This basically follows from the Liebnitz rule, and this is sort of like multiplication of functions on a manifold. So D^q does not satisfy the Liebnitz rule, and instead it gives you, you can write in a nice form after two lines of computation, you can say this is $(D^q f)g(qx) + f(q^{-1}x)(D^q g)$. You may think that this is a minor problem. It turns out not to be so minor. If you prefer, well, I won't.

You should expect, in the quantum group place, that Δe, the action of e on the tensor product, will be like $e \otimes q^E + q^F \otimes e$ where E and F are some vector fields. This is what you should expect. It takes some effort and luck to guess the correct formula, but the end result is the following theorem.

Theorem 2 *Let V, W be representations of $U_q\mathfrak{sl}_2$. Define the action of e, f, h on $V \otimes W$ as $e \to e \otimes q^{h/2} + q^{-h/2} \otimes e$, $f \to f \otimes q^{h/2} + q^{-h/2} \otimes f$, $h \to h \otimes 1 + 1 \otimes h$, or equivalently, $q^h \to q^h \otimes q^h$. Then $V \otimes W$ will be a representation as well.*

The problem is that the relations I have written about commutators are canonical by now. The comultiplication, unfortunately, gives you some freedom. There are many equivalent versions possible, but different books use different ones. I would prefer to use a different version, equivalent by change of variables. Here it gets a little confusing. $\Delta e = e \otimes q^h + q \otimes e$, $\Delta f = f \otimes 1 + q^{-h} \otimes f$. There is a unique coproduct up to equivalence for a general quantum group.

[What happens when q is a root of unity?]

I never said what q was, you can say it is a formal variable. You can say $q = e^\hbar$ or do this in Laurent polynomials over q. For now q will not be a number.

The theorem about finite dimensional irreducible representations will not hold for q a root of unity. There are artificial and meaningless examples where you don't enforce diagonalizability conditions.

So I was saying that the modified Liebnitz rule suggests that you have something similar for the quantum group, and with a little bit of effort you can check this for the rules I've defined above.

This operation Δ is called the coproduct; such a structure is called, if you add a couple more things that I will add later, is called a Hopf algebra. So $U_q\mathfrak{g}$ is a special kind of Hopf algebra. But that's not the place to start.

Begin with the simplest representation, $\mathbb{C}^2 = \langle v_+, v_- \rangle$, where e is the raising operator, f is the lowering operator, and h takes $v_\pm$ to $\pm v_\pm$. The picture is

$$v_- @//^@< -[r]^[1] \longrightarrow v_+$$
$$>$$

but $[1] = 1$ so we get nothing new. What about $\mathbb{C}^2 \otimes \mathbb{C}^2$? Let me make a picture. There is a basis $v_+ \otimes v_+$ and so on.

$$\begin{tikzcd}
& v_+ \otimes v_- \arrow[dl, shift left] \arrow[dr, shift left] & \\
v_- \otimes v_- \arrow[ur, shift left] \arrow[dr, shift left] & & v_+ \otimes v_+ \arrow[ul, shift left] \arrow[dl, shift left] \\
& v_- \otimes v_+ \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

Now:

$$e(v_+ \otimes v_+) = ev_+ \otimes q^h v_+ v_+ \otimes ev_+ = 0;$$
$$e(v_+ \otimes v_-) = v_+ \otimes \dots$$

So there is a minor deformation.

Now, it is known that $\mathbb{C}^2 \otimes \mathbb{C}^2$ decomposes as $\mathbb{C} \oplus V_2$. This is for $\mathfrak{sl}(2)$. You know that applying e to the linear combination of these gives zero. So for $\mathfrak{sl}_2$ it is $\mathbb{C}\langle v_+ \otimes v_- - v_- \otimes v_+ \rangle$. For $U_q\mathfrak{sl}_2$, we get no general theory to say whether this splits. However, here it is possible, $\mathbb{C}^2 \otimes \mathbb{C}^2$ splits off a trivial factor. It is $\mathbb{C}\langle v_+ \otimes v_- - q^{-1} v_- \otimes v_+ \rangle$. Here there is one vector killed by all of these. This is invariant because e, f, and h act by zero. Then you want something else, which includes the vectors $v_+ \otimes v_+, v_- \otimes v_-$, and their image under the action of e and f. This is $v_- \otimes v_+ + q^{-1} v_+ \otimes v_-$. I claim that this is a direct sum, which is not difficult to check.

This is a general result, assuming that q is not a root of unity.

Theorem 3 *1. Let V_n be the irreducible representation of $U_q\mathfrak{sl}_2$ with highest weight n. Then $V_n \otimes V_m \cong \oplus c^k_{nm} V_k$ where $c^k_{nm} \in \mathbb{Z}_+$ are the same as for $\mathfrak{sl}_2$.*

2. The formulas relating the bases in the tensor product and the sum are obtained from the formulas for $\mathfrak{sl}_2$ by insterting factors q^* and replacing $\binom{n}{k}$ by $\begin{bmatrix} n \\ k \end{bmatrix}$, defined as $\frac{[n]!}{[k]![n-k]!}$. I leave you to guess the q-factorial. The multiplicities do not change.

So it's not an obvious thing. The cases $q = 1$ and $q \neq 1$, one might imagine, would be very different, but this theorem says this is not so.

We talked about the tensor product of representations. There is one more thing you can do, and that is deal with duals. If V is a representation then there is a unique way to define the structure of a representation on V^* so that the natural pairing $\langle\ ,\ \rangle : V^* \otimes V \to \mathbb{C}$ is a morphism of representations, i.e., commutes with the action of $U_q\mathfrak{sl}_2$. This is the last missing piece to say that you need multiplication, comultiplication, and something else, this is the something else.

$\mathbb{C}^2$ has the basis v_+, v_-. This has dual basis v^+, v^-. How do you define a representation here? How do you do ev^+? You know that $e\langle v^+, x\rangle = 0$, but you also know that this is $\langle ev^+, q^h x\rangle + \langle v_+, ex\rangle$, so that $\langle ev^+, q^h x\rangle = -\langle v_+, ex\rangle$.

This gives the rule $\langle ev^+, v\rangle = \langle v^+, eq^{-h} v\rangle$. This determines things. You still need to check that the so-defined e, f, and h satisfy the relations. You can go through by hand, but better is to show that it follows by properties of the multiplication and comultiplication.

My time is up, so let me leave an exercise:

Exercise 1 *1. compute explicitly how $U_q\mathfrak{sl}_2$ acts on $(\mathbb{C}^2)^*$.*

2. Will we get the same if we require that the pairing $V \otimes V^* \to \mathbb{C}$ is a morphism? This is the same formula,

Let me stop here. I kept this elementary as possible. Next time I'll talk about the R-matrix, which tells you where things start getting interesting. The last talk may be for general Lie algebras.