Physics Seminar
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I guess I'll just start. Last time I reviewed representation theory of $GL(n,\mathbb{C})$. Last time we saw that irreducible representations have a dominant lowest weight. Now we're going to prove that every weight has a corresponding irreducible representation.

So the idea is to take holomorphic sections of the bundle $(G/p, E)$ which will be representations of G. So for example $\Gamma(\mathbb{CP}^1, \mathscr{O}(k)) = Sym^k\mathbb{C}^2 = (0,k)$. Here $\mathbb{CP}^1$ is $GL(2,\mathbb{C})/\{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix})\}$

Consequences:

1. E will come from an irreducible representation of P so we can build irreducible representations of Lie groups inductively.
2. You can use this result to calculate all cohomology of vector bundles on $\mathbb{CP}^n$, Grassmannians, etc.

To start off, let $G/P = GL(n+1,\mathbb{C})/P \cong \mathbb{CP}^n$. Here $P = \begin{pmatrix} \zeta & * & \cdots & * \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$ for $\zeta \in \mathbb{C}^*$ and $A \in GL(n,\mathbb{C})$.

Definition 1 *A vector bundle E on G/P is homogeneous if G acts on the total space compatibly with the action on G/P and linearly on the fibers, so this diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ G/P \arrow[r, "g"] & G/P \end{tikzcd}$$

Note: Supppose that $g \in P$ and $e \in G/P$ is fixed by g. For example, take $e = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then g will fix E so that if this is commutative, this takes the fiber E_e to itself. In other words, $V = E_e$ is a P-representation. In fact, this determines the homogeneous bundle.

Lemma 1 *The P-representation V uniquely determines E.*

Define $E = G \times_P V$, where the P-action here has $G \times V/(g, pv) \sim (gp, v)$. This has a natural projection to G/P. The idea basically is that you know how P acts on fibers, and because the diagram is commutative, and given an element of P, you know how it acts on fibers. Given another element of G, it will just take a fiber to another fiber.

Consequently we're going to write $E = \mathscr{O}(V)$. The first sort of crucial point is that if you take holomorphic sections $\Gamma(G/P, \mathscr{O}(V))$ it's a G-representation. A section moves up to E, but G acts on this total space, taking a section to another section. This is now a finite dimensional vector space.

I'm going to sort of reformulate this space.

Lemma 2 $\Gamma(G/P, \mathscr{O}(V)) \cong \{\phi : G \to V$ *such that* $\phi(gp) = p^{-1}\phi(g)$ *for all* $p \in P\}$. *Here P is called the "parabolic subgroup."*

The proof uses the description of $E = G \times_P V$. A section s from G/P looks like $s(gP) = [g, \phi(g)]$, where $\phi(g) \in V$. I claim that this is going to be a map from G to V satisfying the relation of the lemma.

Since $s(gpP) = [gp, \phi(gp)] \in G \times V/P$, this is the same as $[g, p\phi(gp)]$. Therefore, $p\phi(gp) = \phi(g)$, so we're done.

The argument works backward as well. It's fairly easy to describe the action now. G acts by $(g\phi)(h) = \phi(g^{-1}h)$.

This way of getting a G-representation from a representation of P is called holomorphic induction.

There doesn't seem to be any geometry any more at the level of this last lemma. As an exercise, take $\mathbb{Z}_3 \subset \mathfrak{S}_3$. Use a representation of $P = \mathbb{Z}_3$ to induce a representation of $G = \mathfrak{S}_3$ in this way.

Define an irreducible representation $(a|b, c, \ldots, d)$ of $P = \left(\begin{array}{c|ccc} \zeta & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right)$ by taking the irreducible representation $(b, c, \ldots, d)$ of $GL(n, \mathbb{C})$ and letting ζ act by ζ^a, and $*$ act trivially.

I'm assuming here that this is ordered $b \le c \le \cdots \le d$.

It turns out that all irreducible representations of P look like this; we're not using that fact. It's obvious that this is irreducible because the $b, \ldots d$ part is.

This gives a homogeneous vector bundle $\mathscr{O}(a|b, c, \ldots, d)$ on $\mathbb{CP}^n = GL(n+1, \mathbb{C})/P$.

Theorem 1 *(Borel-Weil)*
$\Gamma(\mathbb{CP}^n, \mathscr{O}(a|b, c, \ldots, d))$ is the irreducible representation $(a, b, c, \ldots, d)$ of $GL(n+1, \mathbb{C})$ if $a \le b$ and $\{0\}$ otherwise.

Any questions? *Proof.* So basically, most of the work goes into showing this to be an irreducible representation. So we want to use the description of this representation as sections with this property.

Okay, so we know there's a lowest weight ϕ_0 since this is finite dimensional. Think in terms of these functions $GL(n+1, \mathbb{C}) \to (a|b, c, \ldots, d)$ so this is a lowest weight vector of Γ. We want to take this and get a lowest weight vector of P. Let $v_0 = \phi_0(1) \in (a|b, c, \ldots, d)$.

The first claim is that v_0 determines ϕ_0 uniquely. To see this, let $g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \zeta_1 & & & \\ \vdots & & 1 & \\ \zeta_n & & & \end{pmatrix} \in GL(n+1, \mathbb{C})$ and $g^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ zeta_1 & & & \\ \vdots & & 1 & \\ zeta_n & & & \end{pmatrix}$.

Then $\phi_0(g) = \phi_0(g1) = g^{-1}\phi_0(1)$ by the definition of the G-action on Γ, but $g^{-1}\phi_0 = \phi_0$ since ϕ_0 is a lowest weight vector and g^{-1} the the exponential of lowering operators (is the identity plus a strictly lower triangular).

[Why can you let g have this special form?]

We're not done yet. We have $\phi_0(g) = \phi_0(1) = v_0$. Now as we vary $\zeta_1, \ldots, \zeta_n$, then the cosets gP is an affine chart in $\mathbb{CP}^n$.

In particular, this is dense, and just sort of misses out a divisor. So $\phi_0(gp)$ has to satisfy the property so that this is $p^{-1}\phi_0(g) = p^{-1}v_0$. This determines ϕ_0 uniquely. The idea is that as you vary the p the gp give you a dense set, and the v_0 defines ϕ_0 densely. Then the fact that this is holomorphic lets you extend.

The second claim is that v_0 is a lowest weight vector for the P-representation $(a|b, c, \ldots, d)$. To see this, for $p \in P$ of the form *exp* of a lowering operator, you now have $pv_0 = p\phi_0(1) = \phi_0(1p^{-1}) = \phi_0(p^{-1}1) = (p\phi_0)(1) = \phi_0(1)$, since $p \in P$ is also an exponential of a lowering operator for the larger group G. Then since ϕ_0 is a lowest weight it is fixed by the exponential of a lowering operator.

Claim two implies that v_0 is unique up to scale because $(a|b, c, \dots, d)$ is irreducible. Claim one then implies that ϕ_0 is unique up to scale, so Γ is irreducible (or zero). I wasn't going to do that part of the proof. I'll say how you prove that later. The way to show that this is nonzero, it's a little bit complicated but basically uses an induction on a sum $a+b+\cdots+d$. You can make a zero by subtracting off things, tensoring with the determinant representation. You get a short exact sequence of vector bundles on the space, and then move to a long exact sequence. You have to use a special case of something to decompose tensor products; it just takes a while.

I won't say any more about the zero problem. The final part is just to say we get the right lowest weight. Say $p \in D \subset P \subset G$ (the diagonal subgroup). Then $(p\phi_0)(1)$, as above, is pv_0. This will be $\zeta_1^a \zeta_2^b \dots \zeta_{n+1}^d v_0$, where the ζ_i are the diagonal entries. This is $\zeta_1^a \zeta_2^b \dots \zeta_{n+1}^d \phi_0(1)$. So in particular ϕ_0 has weight $a, b, c, \dots, d$. So $p\phi_0$ is $\lambda\phi_0$ and we can tell which it is by looking at a point.

All right, so let me do some examples.

I claimed that all tensor bundles can be written [...]. So $\mathbb{C}^{n+1}$ restricted to P is not irreducible, so $\langle e_1 \rangle$ is preserved. Then $0 \to (1|0, \dots, 0)$, this is an irreducible one dimensional representation of P.

The short exact sequence is

$$0 \to (1|0, \dots, 0) \to (0, \dots, 0, 1)|_P \to (0|0, \dots, 0, 1) \to 0.$$

Then you can take the corresponding sequence of vector bundles, which is just the Euler sequence

$$0 \to \langle \zeta \rangle \to \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}/\langle \zeta \rangle \to 0$$

over $[\zeta] \in \mathbb{CP}^n$.

This first line bundle is $\mathscr{O}(-1) = \mathscr{O}(1|0, \dots, 0)$. As you probably know, the quotient is $T \otimes \mathscr{O}(-1)$, basically the tangent bundle. So $T = \mathscr{O}(-1|0, \dots, 0, 1)$ and $T^* = (1|-1, 0, \dots, 0)$. For example, the Borel-Weil theorem now says $\Gamma(\mathbb{CP}^n, T) = (-1, 0, \dots, 0, 1)$ are the trace-free endomorphisms $End_0(\mathbb{C}^{n+1})$.

You can work out other examples like $\Omega^k = \bigwedge^k T^* = \mathscr{O}(k| -1, \dots, -1, 0, \dots, 0)$. If you want to twist by the hyperplane line bundle, well, $\Omega^k(m) = \mathscr{O}(k-m| -1, \dots, -1, 0, \dots, 0)$. Then $\Gamma(\mathbb{CP}^n, \Omega^k(m))$ is $(k-m| -1, \dots, -1, 0, \dots, 0)$ for $m \geq k+1$ and is otherwise zero.

For higher cohomology, on $\mathbb{CP}^1$ you have Serre duality which tells you $H^1(\mathbb{CP}^1, \mathscr{O}(a|b)) = H^0(\mathbb{CP}^1, \mathscr{O}(1|-1) \otimes \mathscr{O}(-a|-b))^\vee$.

[I stop taking notes because things have become unintelligible.]

Theorem 2 *(Bott-Borel-Weil)*
$H^r(\mathbb{CP}^n, \mathscr{O}(a|b, \dots, c, d, \dots, e))$ where there are r from b to c, is $(b+1, \dots, c+1, a-r, d, \dots, e)$ if dominant, otherwise zero.

Corollary 1 *For an irreducible homogeneous vector bundle E, $H^r(\mathbb{CP}^n, E)$ is nonzero for at most one value of r.*

Flag varieties: Let $1 \leq i < j < \cdots < k \leq n$. Then $\mathbb{F}_{i,j,\ldots,k}(\mathbb{C}^{n+1})$ is defined to be $\{(L_i, L_j, \ldots, L_k)\}$ where L_p is a p-dimensional subspace of $\mathbb{C}^{n+1}$ and $0 \subset L_i \subset L_j \subset \cdots \subset L_k \subset \mathbb{C}^{n+1}$. So for example, $\mathbb{CP}^n = \mathbb{F}_1,\ Gr_k(\mathbb{C}^{n+1}) = F_k$, a complete flag is $\mathbb{F}_{1,2,\ldots,n}$.

So in this case we have $GL(n+1, \mathbb{C})/P = \begin{pmatrix} A \in GL(i,\mathbb{C}) & * & * \\ & B \in GL(j-i,\mathbb{C}) & * \\ 0 & & \ddots \end{pmatrix}$. There exists a homogeneous vector bundle $\mathscr{O}(a, \ldots, b|c, \ldots, d|\cdots)$ on $F_{i,j,\ldots,k}$ where there are i entries from a to b. So for example $\mathscr{O}(a|b|c|\cdots|d)$ on $\mathbb{F}_{1,2,\ldots,n}$.

Theorem 3 $\Gamma(\mathbb{F}, \mathscr{O}(a, \ldots, b|c, \ldots, d|\cdots))$ *is* $(a, \ldots, b, c, \ldots, d, \ldots)$ *if dominant, otherwise zero.*

I have a paper on this, you can refer to Bott's paper in the Annals in the 50s, and there's a similar proof in [...]

[Let me make one final comment, that this is just one case of a larger theory. It turns out that you can get, if you want, things like Verma modules, which are not finite dimensional, but you can get them as global sections on some appropriate homogeneous sheaf on a flag variety.]

I want to hear more about quantum cohomology on a flag variety. I am less interested in talking about quantum groups, but since I'm not the only one here, maybe we should ask other people. Everyone but me wants me to give a talk on quantum groups.