Physics Seminar
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Last time, let me remind you of Gabriel's theorem.

1. A quiver Q is finite type if and only if its underlying graph $\hat{Q}$ has type $A_n,\ D_n,\ E_6,\ E_7$, or E_8. This means it has finitely many indecomposable representations.
2. If Q is of finite type, then there is a bijection between positive roots of the corresponding Lie algebra and indecomposable representations.

In particular this says how many representations it has. So for A_n this is $\frac{n(n+1)}{2}$; for D_n it is $n^2 - n$.

But this is stronger because it gives a bijection. For an indecomposable representation, you can just take the dimension vector. To go the other way is a little more tricky.

So let's go on. We'll start with a definition and some notation. We start with a Coxeter functor.

Definition 1 *Say $x \in Q_0$, the vertices of the quiver Q. We can construct the quiver $S_x(Q)$, which is the same quiver but with all arrows incident on x reversed.*

Now let $x \in Q_0$ be a sink. Define C_x^+, a functor from representations of Q to representations of $S_x(Q)$.

So say I have a sink x, I want to show how to define the functor. The vertex x, instead of being enriched with the same vector space $V(x)$, is enriched by the kernel of the map $V_1 \oplus \cdots \oplus V_k \to V(x)$ with the maps the projections.

If I have a source x I can define C_x^- similarly by replacing $V(x)$ with coker$(V(x) \to V_1 \oplus \cdots \oplus V_k)$.

One more definition. Consider the simple representation where you have k in one place and zero elsewhere. This is represented by the symbol ϵ_x, where x is the vertex. We denote the corresponding reflection by S_x. This is defined to be $S_x(\beta) = \beta - (\epsilon_x, \beta)\epsilon_x)$, where $(,\ )$ is the symmetric Euler form so $(\epsilon_x, \beta) = \langle \epsilon_x, \beta \rangle + \langle \beta, \epsilon_x \rangle$.

Theorem 1 *Let Q be a quiver of finite type and V an indecomposable representation of Q. Say x is a sink. Then one of the following is true:*

1. *$V = E_x$, and $C_x^+ V$ is a representation of $S_x Q$ which is* 0.
2. *$C_x^+ V$ is indecomposable and the dimension vector of $C_x^+ V$ will be S_x applied to the original dimension vector. Alsoo $C_x^- C_x^+ V \sim V$.*

These representations are not equivalent, but there is a nice correspondence between the indecomposables one and the other.

Before I prove this one, let me write down some corollaries. I can number my vertices so that arrows go in increasing order.

Then 1 is a source and n is a sink. Then $C_n^- \circ \cdots \circ C_2^- \circ C_1^-$ is a functor from $Rep(Q)$ to $Rep(S_n \cdots S_2 S_1 Q) = Rep(Q)$. Similarly, $C_1^+ \cdots C_{n-1}^+ C_n^+$ is a functor from $Rep(Q)$ to itself. Denote these by C^- and C^+ with corresponding reflections c^- and c^+ or just c.

Corollary 1 *V is an indecomposable representation then $C^+ V = 0$ or is indecomposable with dimension vector $d_{C^+(V)} = c^+(d_V)$.*

Lemma 1 *Let Q be a quiver where $\hat{Q} \in \{A_n, D_n, E_6, E_7, E_8\}$ and α a dimension vector. Then $c^k \alpha$ is a negative vector.*

So c has finite order m. Then consider $\beta = \sum_0^{m-1} c^i \alpha$. Then $c\beta = \beta$, so that $\beta = 0$. This is because $\langle \alpha, c\beta \rangle = -\langle \beta, \alpha \rangle$. So if $c\beta = \beta$ then $(\beta, \beta) = 2\langle \beta, \beta \rangle = \langle \beta, \beta \rangle + \langle \beta, c\beta \rangle = 0$.

So next, keep the same assumptions and let V be an indecomposable representation of the quiver. I need to show that d_V is a positive root. From the corollary we have that $d_{(C^+)^k V} = c^k d_V$. Then there exists a k for which this is a negative vector. So some power of it must give zero.

So consider $W = (C^+)^{k-1} V$. We know that $C^+(W) = 0$. But this is $C_1^+ \cdots C_n^+ W$. So then at some point I will get E_l for some l. Then $W = C_n^- \cdots C_{l+1}^- E_l$. Then $V = (C^-)^{k-1} C_n^- \cdots C_{l+1}^- E_l$ so that $d_V = (c^-)^{k-1} s_n \cdots s_{l+1} \epsilon_l$. This makes this root positive.

This is one direction. Now we start with a positive root and want to show that it is a dimension vector. I'll go over an example.

[What is going on is, we want to get all indecomposables. We only know of the E_i to start. So we apply reflections to get more, but we can only use the reflections when these are sinks or sources.]

Now for the other direction we start with a positive root α. From the lemma we can choose a minimal k such that $c^{k+1}\alpha$ is negative. Consider $\beta = c^k\alpha$. So then $s_1 \cdots s_n\beta$ is negative. Now s_i permutes positive roots except for ϵ_i. So then $s_{l+1} \cdots s_n\beta = \epsilon_l$, so $\beta = s_n \cdots s_{l+1}\epsilon_l$. Then $C_n^- \cdots C_{l+1}^- E_l$ is an indecomposable with dimenstion vector β; then $(C^-)^k C_n^- \cdots C_{l+1}^- E_l$ is an indeocomposable with dimension vector α.

Consider the indecomposable

$$k \xrightarrow{1} k \xrightarrow{1} k$$

with corresponding dimension vector $e_1 - e_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

How do we go the other way? We have

$$s_1\alpha = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \left[\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \right] \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

So $s_2 s_1\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \epsilon_3$.

Now $s_1 A_3$ looks like

$$0 \longleftarrow k \longrightarrow k$$

while $s_2 s_1 A_3$ is

$$0 \longrightarrow 0 \longleftarrow k$$

Now if I apply C_2^+ I get

$$0 \xleftarrow[1]{} k \longrightarrow k$$

and then C_1^+ to get

$$k \longleftarrow k \longrightarrow k$$

Let's look at a trickier example,

$$\begin{tikzcd} k \arrow[r] & k^2 & k \arrow[l] \\ & k \arrow[u] & \end{tikzcd}$$

with dimension vector $\alpha = e_2 + e_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ and $s_3 s_2 s_1 \alpha = \alpha$ and $s_4 s_3 s_2 s_1 \alpha = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Finally you get $s_3 s_2 s_1 s_4 s_3 s_2 s_1 \alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ so that $\alpha = s_1 s_2 s_3 s_4 s_1 s_2 s_3 \epsilon_4$.

So you can start with E_4 in the quiver

$$\begin{tikzcd} 0 & k \arrow[l] \arrow[r] \arrow[d] & 0 \\ & 0 & \end{tikzcd}$$

and apply C_3^+ to get

$$\begin{tikzcd} 0 & k \arrow[l] \arrow[d] & k \arrow[l, swap, "1"] \\ & 0 & \end{tikzcd}$$

and then C_2^+ yields

$$\begin{tikzcd} 0 & k \arrow[l] & k \arrow[l, swap, "1"] \\ & k \arrow[u, "1"] & \end{tikzcd}$$

and then C_1^+ :

$$\begin{tikzcd} k \arrow[r, "1"] & k & k \arrow[l, swap, "1"] \\ & k \arrow[u, "1"] & \end{tikzcd}$$

Here it gets a little tricky because I have a kernel. At the end I get

$$\begin{tikzcd} \{(\beta,\gamma)|\beta+\gamma=0\} \arrow[r] & R = \{(\alpha,\beta,\gamma)|\alpha+\beta_\gamma=0\} & \{(\alpha,\beta)|\alpha+\beta=0\} \arrow[l] \\ & \{(\alpha,\gamma)|\alpha+\gamma=0\} \arrow[u] & \end{tikzcd}$$

Why does the Coxeter functor applied to an indecomposable give either 0 or an indecomposable?

The idea is as follows. Say $x \in Q_0$ is a sink. Then I have $C_x^+ : Rep(Q) \to Rep(S_x Q)$. I can also go in the other direction with C_x^-.

First we'll construct a natural transformation i_x from $C_x^- C_x^+$ to the identity. In a similar way we construct a functor p_x from the identity to $C_x^+ C_x^-$. Then the proof depends on some characteristics of these natural transformations.

$C_x^- C_x^+$ at x is the cokernel of the projection from the kernel of the map into $V(x)$. This is the quotient of the sum $\oplus V_i$ by the kernel, which is the original space. This is pure linear algebra.

There are some properties of these two natural transformations.

1. If i_x is an isomorphism then $d_{C+xV} = s_x d_V$ and similarly for the other one.
2. If $V = C_x^- W$ then i_x is an isomorphism.
3. if $x \in Q_0$ is a sink then $V = C_x^- C_x^+ \oplus \tilde{V}$, the cokernel of i_x.

The proof of all of this is not hard. Let me skip most of it. Let V be an indecomposable representation. If $V = E_x$ then $C_x^+ V = 0$. If V is not this, then we will show that $C_x^+ V$ is indecomposable.

Since V is indecomposable, either $V = \tilde{V}$ or $V = C_x^- C_x^+ V$. If $V = \tilde{V}$ then $V = E_x$ because it's concentrated at x. Otherwise $V = C_x^- C_x^+ V$. We want to show that $C_x^+ V$ is indecomposable. Say it can be written $W_1 \oplus W_2$. Then C_x^- preserves the direct sum so this is $C_x^- W_1 \oplus C_x^- W_2$. Since V is indecomposable, now one of these has to be zero. Then apply C_x^+ to that piece to get $C_x^+ C_x^- W_2 = 0$. But we have p_x an isomorphism from the identity to $C_x^+ C_x^-$. This implies $W_2 = 0$, so that $C_x^+ V$ is indecomposable, as desired.

[Next week Justin will be talking about ...]

I have an article about this at s̃awon/borel_weil.ps.gz

This is the most important thing about geometric representations. He'll be talking about projective spaces.

This allows you to calculate cohomology groups of vector bundles etc.