Infinite Dimensional Lie Algebras
March 9, 2005

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March 18, 2005

Okay. Meanwhile, let me make a picture that I will use later.

[It is some vertical and diagonal lines.]

The picture is not a great success. You might guess what it is supposed to be. We were discussing affine root systems and affine Lie algebras. Today we will talk only about root systems.

$\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$. We have $\hat{R} = \hat{R}_{re} \cup \hat{R}_{im}$ where these are $\alpha + n\delta$ for $\alpha \in R$ and $n \in \mathbb{Z}$ and $n\delta$ for $n \in \mathbb{Z}\backslash\{0\}$.

There is a reflection again $s_{\hat{\alpha}} : \hat{\mathfrak{h}}^* \to \hat{\mathfrak{h}}^*$ by the usual, $\hat{\lambda} \mapsto \hat{\lambda} - \frac{2(\hat{\lambda},\hat{\alpha})}{(\hat{\alpha},\hat{\alpha})}\hat{\alpha}$.

Last time we did $\mathfrak{sl}(2)$; today we'll work in general. If $\hat{\alpha} = \alpha + n\delta$ and $\hat{\lambda} = \lambda + k\Lambda_0 + a\delta$ then $(\hat{\lambda}, \hat{\alpha}) = (\alpha, \lambda) + nk$. Then

$$s_{\hat{\alpha}} : \hat{\lambda} \mapsto \hat{\lambda} - \frac{2(\hat{\lambda}, \hat{\alpha})}{(\alpha, \alpha)}(\alpha + n\delta) = (\lambda - \frac{2(\alpha, \lambda)}{\alpha, \alpha}\alpha - \frac{2}{(\alpha, \alpha)}nk\alpha) + k\Lambda_0 + (a - \frac{2(\alpha, \lambda)}{\alpha, \alpha}n - \frac{2}{\alpha, \alpha}nk)\delta$$

$$= s_\alpha(\lambda) - nk\alpha^\checkmark + k\Lambda_0 + (a - (\lambda, \alpha^\checkmark)n - \frac{(\alpha^\checkmark, \alpha^\checkmark)}{2}nk)\delta,$$

where $\frac{2\alpha}{(\alpha,\alpha)} = \alpha^\checkmark$.

Note that this preserves $\hat{\mathfrak{h}}^*_k = \mathfrak{h}^* \oplus \mathbb{C}\delta + k\Lambda_0$, the level.

As a special case, $\hat{\mathfrak{h}}^*_0 = \mathfrak{h}^* \oplus \mathbb{C}\delta$ and $s_{\hat{\alpha}}$ sends $\lambda + a\delta \mapsto s_\alpha(\lambda) + (a - (\lambda, \alpha^\checkmark)n)\delta$.

The other case ignores the δ part. Say we have $\hat{\mathfrak{h}}^*_k/\mathbb{C}\delta \cong \mathfrak{h}^*$ (as a set). Then $s_\alpha : \lambda + k\Delta_0 \bmod \delta \mapsto s_\alpha(\lambda) - nk\alpha^\checkmark + k\Lambda_0$.

So in the first case it is the composition of the usual reflection and a shear; in the second the reflection and a translation. This is not a linear but affine action.

It would be interesting to try to seperate the reflection from the translations, or isolate the shear. It's not actually that difficult to do this.

Let me define for $\alpha \in R$, $\tau_{\alpha^\vee} = s_\alpha \circ s_{-\alpha+\delta}$. The point is that except for δ this would give the same reflection. The composition should then be the identity in the finite part.

It doesn't take long to do the computation in full generality, so let me just do it.

It maps

$$\hat{\lambda} \mapsto (\lambda + k\alpha^\vee) + k\Lambda_0 + (a - (\lambda, \alpha^\vee) - \frac{(\alpha^\vee, \alpha^\vee)}{2} k)\delta.$$

If I look at the level zero space $\hat{\mathfrak{h}}_0^*$ I get $(\lambda + a\delta) \mapsto \lambda + (a - (\lambda, \alpha^\vee))\delta$, a pure shear.

In the other case I get $\lambda + k\Lambda_0 \mod \delta \mapsto \lambda + k\alpha^\vee + k\Lambda_0 \mod \delta$.

I did not actually do the calculation myself, but I think you can do it easily.

Let me now define $\tau_\mu : \hat{\mathfrak{h}}^* \to \hat{\mathfrak{h}}^*$ by $s_\mu \circ s_{-\mu+\delta}$.

Now I can form the main theorem.

Theorem 1 *1.* $\tau_{\mu_1+\mu_2} = \tau_{mu_1}\tau_{\mu_2}$.

2. $\hat{W} = W \ltimes \tau(Q^\vee)$*, where* $\hat{W}$ *is the affine Weyl group and* $Q^\vee$ *is the lattice generated by* $\alpha^s urd$

Proof.

1. Explicit computation

2. Obviously, we can check that $\tau(Q^\vee) \subset \hat{W}$. This is because we can write $\mu = n_1\alpha_1^\vee + \cdots$, and then $\tau_\mu = \tau_{\alpha_1^\vee} \circ \tau_{\alpha_1^\vee} \circ \cdots$.

   It is also obvious that $W \subset \hat{W}$. It remains to show two things. First we must show that everything in $\hat{W}$ can be written as a product of the two of these, second we must show it uniquely.

   It suffices to show that the generators can be written in this form. $s_{\alpha+n\delta} = s_\alpha \circ \tau_{\pm n\alpha^\vee}$.

   We know that the generators can be written $w\tau_\mu$ where $w \in W$ and $\tau_\mu \in \tau(Q^\vee)$; then the claim is that you can write a product of these like this. You write $(w_1\tau_{\mu_1})(w_2\tau_{\mu_2}) = w_1w_2(w_2^{-1}\tau_{\mu_1}w_2)\tau_{\mu_2}$, and this in the middle is $\tau_{w_2^{-1}(\mu_1)}$.

   I leave it to you to check that if $w_1\tau_{\mu_1} = w_2\tau_{\mu_2}$.

So this is a funny thing, it contains this big abelian subgroup, this lattice. Let me show what it looks like in the case of $\mathfrak{sl}(2)$ and maybe $\mathfrak{sl}(3)$, and let me concentrate on the special case, only worried about how the whole thing acts on the space. Then it has a very nice geometric interpretation.

The action of $\hat{W}$ on $(\hat{\mathfrak{h}}_k^*)^{\mathbb{R}}/\mathbb{R}\delta \cong \eqsim^{*\mathbb{R}}$ in particular has easy geometric meaning.

Let me write $s_{-\alpha+n\delta} : \lambda \to s_\alpha(\lambda) + nk\alpha^{\checkmark}$ and $\tau_{\alpha^{\checkmark}} : \lambda \to \lambda + k\alpha^{\checkmark}$.

The $s_{-\alpha+n\delta}$, I claim, is still a reflection.

What is $s^2_{-\alpha+n\delta}$? It is $s^2_\alpha(\lambda + s_\alpha(nk\alpha^{\checkmark}) + nk\alpha^{\checkmark}$ and I am left with λ since s_α acts on $\alpha^{\checkmark}$ by negating it.

So what is fixed by $s_{-\alpha+n\delta}$? I need $\lambda - (\lambda, \alpha^{\checkmark}) + nk\alpha^{\checkmark} = \lambda$, and moving things around I get $(\lambda, \alpha) = nk$. So the group, the special part of the action of this group is generated by reflections through the hyperplane generated by this. The formulas work in full generality; I worked over $\mathbb{R}$ so that these are honest "orthogonal reflections."

In other words the action of $\hat{W}$ in this vector space is generated by reflections with respect to $H_{\alpha,nk}$. What do we get for $\mathfrak{sl}(2)$? There we have only one positive root so we have a one dimensional real space.

Example 1 $\widehat{\mathfrak{sl}}(2)$ *for* $k = 1$. *I have a line marked by multiples of* $\alpha/2$, *since* $(\alpha/2, \alpha) = 1$. *The hyperplanes are just points, so my affine Weyl group (its action) is generated by reflections through these points. What is the group generated by these? It's a semidirect product of the usual Weyl group* ($\mathbb{Z}/2$) *and translations. The translation length is twice the difference between the reflection points, namely* α.

What will it be for $\mathfrak{sl}(3)$? I've drawn (poorly) the hyperplanes and roots for $\mathfrak{sl}(3)$; using a computer I can draw it perfectly. It's reflections around affine hyperplanes orthogonal to the roots and with affine coefficient an integer multiple of half the length of a root.

The picture or the one for $\mathfrak{sl}(2)$ brings up questions, like what is the fundamental domain? It is not exactly obvious but you can see it is half the length, the distance from one axis point to the next. If you change k you just rescale. For $\mathfrak{sl}(3)$ it is one of the little triangles. It is obvious you can get there but slightly less obvious that it is the fundamental domain. If you have a chain of triangles you pass through, you pull back across things you cross twice, you get that it is the identity. The point is that it is exactly the same as for the normal Weyl group.

Definition 1 *A Weyl alcove is a connected component of* $\mathfrak{h}^{*\mathbb{R}} - \cup H_{\alpha,nk}$

I don't know who came up with the terminology, to go from chamber to alcove. The theorem is

Theorem 2 *The affine Weyl group acts simply transitively on the set of Weyl alcoves.*

This exactly means that each alcove is the fundamental domain for the action of the affine Weyl group. Transitivity is obvious, simplicity is a little more effort, but it generally follows the finite dimensional case.

The important thing is that unlike the finite case, here every alcove is bounded, but you have infinitely many of them. This presents some unusual corollaries.

If you remember, for finite dimensional Lie representations, these are classified by positive roots. You might expect it to be classified by weights in positive alcoves. But Weyl alcoves are always bounded so there are only finitely many weights. So you might expect only finitely many representations. There are no finite dimensional representations. The question is, what is the analog of a finite dimensional representation here.

This whole theory is so nice that we might hope for some analog of it for affine Lie algebras. The question is, what kind of representation are we looking for? Of course we can, and we'll do it in about a week. The theory will be different.

Next time I'll continue with simple roots and generators of the Lie algebra.