Infinite Dimensional Lie Algebras
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Sorry I'm late. Okay, so, anyone keeps track of lecture numbers? So, let me remind you what we are doing. We're studying affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k \oplus \mathbb{C}d$. We have $[xt^n, yt^m] = [x, y]t^{m+n} + n\delta_{n,-m}(x, y)k$. d is the grading operator so it acts on t^m by m. From now on I will have $\mathfrak{g}$ a simple algebra, and $(\ ,\ )$ is normalized by $(\alpha, \alpha) = 2$ for long roots.

The main result last time was $\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus (\oplus_{\hat\alpha\in\hat R} l\hat{a}g_{\hat\alpha})$ where $\hat R = \hat R_{re} \cup \hat R_{im}$ and $\hat R_{re} = \{\alpha + n\delta, \alpha \in R, n \in \mathbb{Z}\}$.

I should have said $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}k \oplus \mathbb{C}d$ and $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\lambda_0 \oplus \mathbb{C}\delta$.

So we have $\hat{\mathfrak{g}}_{\alpha+n\delta} = \mathfrak{g}_\alpha t^n$, and then $\hat R_{im} = \{n\delta, n \neq 0\}$ so that $l\hat{a}g_{n\delta} = \mathfrak{h}t^n$.

The case we discussed was $\widehat{\mathfrak{sl}(2)}$ where you have

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \uparrow \\
f[1] & h[1] & e[1] & \delta \\
f[1] & h[1] & e[1] & \mathfrak{h}^* \longrightarrow \\
f[1] & h[1] & e[1] & \\
\vdots & \vdots & \vdots &
\end{array}$$

Note: $\hat R$ generates a subspace of codimension one in $\hat{\mathfrak{h}}^*$, in that all have zero coefficient in front of λ_0. This is the adjoint representation so the central element acts by zero.

The second point is that $(\,,\,)|_{\mathfrak{h}^* \oplus \mathbb{C}\delta}$ is degenerate with kernel $\mathbb{C}\delta$.

It is positive semidefinite on $\mathfrak{h}^*_{\mathbb{R}} \oplus \mathbb{R}\delta$. It is positive definite on the left summand. This makes life a whole lot different from the finite dimensional case.

Now I'm going to try to develop the same standard theory. We studied the properties of the root system, including symmetries, generating sets, and then move to matrices or Dynkin diagrams.

The first thing would be to see whether the set of roots is invariant under some set of reflections. I'll imitate what we did in the finite dimensional case. Whenever you have a root α, you can find an $\mathfrak{sl}(2)$ triple corresponding to it. Let's see if we can do the same thing here.

Lemma 1 *Let $\hat{\alpha} = \alpha + n\delta \in \hat{R}_{re}$. Then*

1. $(\hat{\alpha}, \hat{\alpha}) \neq 0$.
2. *Choose $e_{\hat{\alpha}} \in \mathfrak{g}_{\hat{\alpha}}, f_{\hat{\alpha}} \in \mathfrak{g}_{-\hat{\alpha}}$ such that $(e_{\hat{\alpha}}, f_{\hat{\alpha}} = \frac{2}{(\alpha,\alpha)}$ and let $h_{\hat{\alpha}} = [e_{al\hat{p}ha}, f_{\hat{\alpha}}]$. Then $e_{\hat{\alpha}}, f_{\hat{\alpha}}, h_{\hat{\alpha}}$ satisfy the relations of $\mathfrak{sl}(2)$.*

Proof.

1. $(\alpha + n\delta, \alpha + n\delta) = (\alpha, \alpha) \neq 0$.
2. We can write $e_{\hat{\alpha}} = e_\alpha t^n, f_{\hat{\alpha}} = f_\alpha t^{-n}$. Then $h_{\hat{\alpha}} = [e_{al\hat{p}ha}, f_{\hat{\alpha}}] = [e_\alpha, f_\alpha] + n\frac{2}{(\alpha,\alpha)}k = h_\alpha + n\frac{2}{(\alpha,\alpha)}k$.

   If I commute with $e_{\hat{\alpha}}$ or $f_{\hat{\alpha}}$ the second term gives zero. So we get $[h_{\hat{\alpha}}, e_{\hat{\alpha}}] = [h_\alpha, e_{\hat{\alpha}}] = 2e_{\hat{\alpha}}$. Similarly check the relation for f and you get the same.

By the way, this fails if you take an imaginary root. It would not give you $\mathfrak{sl}(2)$, but instead the finite dimensional Heisenberg algebra. I'm not going to use that much. Let me think of real roots first.

Lemma 2 *Define for $\hat{\alpha} \in \hat{R}_{re}$, $s_{\hat{\alpha}} : \hat{\mathfrak{h}}^* \to \hat{\mathfrak{h}}^*$ by $s_{\hat{\alpha}}(\hat{\beta}) = \hat{\beta} - \frac{2(\hat{\beta},\hat{\alpha})}{(\hat{\alpha},\hat{\alpha})}\hat{\alpha}$.*

Then $s_{\hat{\alpha}}(\hat{R}) = \hat{R}$.

You should recall that $(\hat{\alpha}, \hat{\alpha})$ is never zero by the last lemma; for imaginary roots this just does not make sense.

I can do this by looking at the actions explicitly since I know what all of my roots are, but I think it is more instructive to modify the finite dimensional proof.

Proof. Consider $\hat{\mathfrak{g}}$ as a module over $\mathfrak{sl}(2)$ generated by $e_{\hat{\alpha}}, f_{\hat{\alpha}}, h_{\hat{\alpha}}$. If I have $x \in \hat{\mathfrak{g}}_{\hat{\beta}}$, if I have a module over $\mathfrak{sl}(2)$ then the weights are symmetric. So x has weight equal to $\langle \hat{\beta}, h_{\hat{\alpha}} \rangle$, and it is an easy calculation, basically one we already did, that this is $\langle \beta, h_\alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

Okay. You remember how the proof goes for the normal finite dimensional case. If you have such a module and you start with weight λ, you can apply f and get weight $-\lambda$. This will be nonzero, so the set of weights is symmetric. Thus $f_{\hat{\alpha}}^n x \neq 0$, $\in \hat{\mathfrak{g}}_{\hat{\beta}} - n\hat{\alpha} = \hat{\mathfrak{g}}_{s_{\hat{\alpha}}(\hat{\beta})}$ for $n = \frac{2(\hat{\beta},\hat{\alpha})}{(\hat{\alpha},\hat{\alpha})}$.

This is the same theory as in finite dimensional Lie algebras, but with a twist. I actually cheated a little bit. Can anyone see where I cheated? I said that in every $\mathfrak{sl}(2)$ module the roots are symmetric. That's not true, it's only true in finite dimensional modules. Unfortunately, $\hat{\mathfrak{g}}$ by itself is not finite dimensional.

So now what I want to do is to split $\hat{\mathfrak{g}}$ into the direct sum of finite dimensional $\mathfrak{sl}(2)$ modules. A simple argument is that the weights add. I slice my original set with slices parallel to whatever my original one is. As long as the slope is finite, each slice will be of finite size. So it is still the direct sum of finite dimensional modules.

The key argument is that this is what people call "locally finite dimensional," so that any piece I take will be in a finite dimensional module, so I can ignore the rest and take what I want from the normal theory of finite dimensional modules over $\mathfrak{sl}(2)$. I should also have said that each of the fractions is an integer. The same thing as before but I replace "finite dimensional" with "locally finite dimensional."

It is interesting to study what kind of group these reflections generate. They are reflections of a very funny nature.

1 Affine Weyl group

Let me start with the example of $\widehat{\mathfrak{sl}(2)}$. So $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$, where $\mathfrak{h}^* = \mathbb{C}\alpha$. Then $\hat{\mathfrak{h}}_0 = \mathfrak{h}^* \oplus \mathbb{C}\delta$.

If I take $\hat{\alpha} = \alpha + n\delta$, then where does this map $\lambda + a\delta$? We have

$$s_{\hat{\alpha}}(\lambda + a\delta = \lambda + a\delta - \frac{2(\lambda + a\delta, \alpha + n\delta)}{(\alpha,\alpha)}(\alpha + n\delta) =$$

$$(\lambda + n\delta) - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}(\alpha + n\delta) = s_\alpha(\lambda) + (a\delta - \frac{2(\lambda,\delta)}{(\alpha,\alpha)}n\delta),$$

and if I normalize so that $(\alpha,\alpha) = 2$ I get something even easier. Since the dimension is one the only reflection in $\mathfrak{h}^*$ sends λ to $-\lambda$. So this is $-\lambda + (a\delta - (\lambda,\alpha)n\delta)$.

For $n = 0$ we have $s_{\hat{\alpha}} : \lambda + a\delta \mapsto -\lambda + a\delta$.
For $n = 1$ it is $s_{\hat{\alpha}} : \lambda + a\delta \mapsto -\lambda + a\delta - (\lambda,\alpha)\delta$.

Can you visualize the two dimensional linear operator at work here? It reflects in one of the directions and then shears.

If we identify $\mathfrak{h}^*$ with $\mathbb{C}$ so that $\alpha \iff 2$ then $s_\alpha : \lambda \mapsto -\lambda$ and $s_{\alpha+\delta} : \lambda + a\delta \mapsto -\lambda + a\delta - \lambda\delta$.

This is a very funny operation. So in the same example as $\mathfrak{sl}(2)$ let's see what kind of group they generate. This is $\hat{W} = W_{aff}$, the affine Weyl group.

Note $\tau = s_\alpha \circ s_{\alpha+\delta} : \lambda + a\delta \mapsto \lambda + a\delta - \lambda\delta$. Then $\tau^2 : \lambda + a\delta \mapsto \lambda + a\delta - 2\lambda\delta$. From here you can see that τ is generating a group isomorphic to $\mathbb{Z}$.

Theorem 1 *For* $\hat{\mathfrak{g}} = \mathfrak{sl}(2)$, $\hat{W} = \mathbb{Z}_2 \ltimes \mathbb{Z}$.

These come from s_α and τ respectively. Let's check the commutation relations to be $s_\alpha \tau s_\alpha = \tau^{-1}$.

Here we have a group which is infinite and generated by two reflections or a reflection and a shear, an element that acts freely. This is different than the usual Weyl groups because it includes shears.

Okay.

So more generally $\hat{\mathfrak{h}}^* = \mathbb{C}\alpha \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ and

$$s_{\alpha+n\delta}(\lambda + k\Lambda_0 + a\delta) = (\lambda + k\Lambda_0 + a\delta) - (\lambda + k\Lambda_0 + a\delta, \alpha + n\delta)(\alpha + n\delta) =$$

$$\lambda + k\Lambda_0 + a\delta - ((\lambda, \alpha) + kn)(\alpha + n\delta) = (\lambda - (\lambda, \alpha)\alpha - kn\alpha) + k\Lambda_0 + (a - ((\lambda, \alpha) + kn))\delta.$$

Let me make some observations and then stop. We have not changed the Λ_0 coefficient. Each of these affine subspaces is preserved by the action. Even in the pure finite dimensional part I don't get the usual reflection. I get the usual reflection but also a $kn\alpha$ term which is not linear in λ, so a translation. Also there is a rather complicated action in the third direction.

Actually this is not really so messy. Let me tell you one thing.

How $\hat{W}$ acts on $\hat{\mathfrak{h}}^*_k = \{\lambda + k\Lambda_0 + a\delta\}$. The statement is that each of these preserves it. How does s_α act? $s_\alpha : \lambda + k\Lambda_0 + a\delta \mapsto -\lambda + k\Lambda_0 + a\delta$. If $n = 1$ I get $s_{\alpha+\delta} : \lambda + k\Lambda_0 + a\delta \mapsto -\lambda + 2k + \cdots$. So on the finite dimensional part, I get that $s_{\alpha+\delta}$ is reflection around k instead of 0. So $\tau : \lambda + \cdots \mapsto \lambda + 2k + \cdots$

So τ generates again a group isomorphic to $\mathbb{Z}$. So either you say it's generated by two reflections, or by a reflection and a translation.

Next time I will discuss this group in full generality. It will include the ordinary Weyl group but also something corresponding to the translations or shears.