# Infinite Dimensional Lie Algebras <br> March 14, 2005 

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Okay, guys, sorry for delay, let me start from where we stopped last time. We're looking at affine root systems, which come from affine Lie algebras. So we have $\hat{R}=\hat{R}_{r e} \cup \hat{R}_{i m}$, where the roots in these are of form $\{\alpha+n \delta\}$ and $\{n \delta\}$. I don't need to remind you of the details.

Remember each root defines a reflection $\hat{\alpha} \in \hat{R}_{r e} \mapsto s_{\hat{\alpha}}: l \hat{a} h^{*} \rightarrow \hat{\mathfrak{h}}^{*}$. These generate a group $\hat{W}=W \ltimes Q^{\sqrt{ }}$.

On the levels we have $\tau_{\alpha \vee}$ acting on $\hat{\mathfrak{h}}_{k}^{*} / \mathbb{C} \delta \cong \hat{\mathfrak{h}}^{*}$ by $\lambda \rightarrow \lambda+k \alpha^{\sqrt{ }}$.
On $\widehat{\mathfrak{h}_{k}^{*}} \mathbb{R}^{\mathbb{R}} / \mathbb{R} \delta \cong\left(\mathfrak{h}^{*}\right)^{\mathbb{R}}, s_{\hat{\alpha}}$ acts by the usual reflection around the hyperplane $H_{\alpha, n k}=\{\lambda \mid(\lambda, \alpha)+$ $n k=0\}$. Up to a change of sign that's the same thing as last time. I should probably keep it the same as last time, $\{\lambda \mid(\lambda, \alpha)=n k\}$.

So $\hat{\alpha}=-\alpha+n \delta$.
In the finite dimensional case you have hyperplanes intersecting at the origin; in the affine case you really have to add more hyperplanes, affine ones parallel to the original ones. The Weyl group acts simply transitively on the sectors carved out by these, called Weyl alcoves.

The next thing to do is to find a way to generate the Weyl group with a small number of roots. For finite dimensional $\mathfrak{g}$ we split $R$ into $R_{+} \cup R_{-} \rightarrow \Pi=\left\{\alpha_{1}, \cdots, \alpha_{r}\right\}$, the simple roots, which give you a Cartan matrix or a Dynkin diagram. Simple reflections generate the Weyl group, and this is just a reminder, let's see if we can do the same thing for an affine system.

Can we split an affine root system into positive and negative roots? Splitting into positive and negative along left and right doesn't work. You know you want $\alpha$ to be positive and $-\alpha$ to be negative. You take everything with positive $\delta$ to be positive, everything with negative $\delta$ to be negative.

| $-\alpha+\delta$ | $\delta$ | $\alpha+\delta$ |
| :---: | :---: | :---: |
| $-\alpha$ | 0 | $\alpha$ |

So if I view $\hat{\alpha}=\alpha+n \delta$ as $\mathfrak{g}_{\hat{\alpha}}=\mathfrak{g}_{\alpha} t^{n}$, then we are splitting off powers of $t$, so that $\oplus_{n \geq 0} \mathfrak{g}_{\hat{\alpha}}=$ $\mathfrak{g} \mathbb{C}[t]$. The other would give you $\mathfrak{g} \mathbb{C}\left[t^{-1}\right]$. We have to split constants according to their sign.

So now, more formally, $\hat{R}_{+}=\left\{\alpha+n \delta \mid n>0\right.$ or $\left.\left(n=0, \alpha \in R_{+}\right)\right\}$and $\hat{R}_{-}=\{\alpha+n \delta \mid n<0\}$ or ( $\left.\left.n=0, \alpha \in R_{-}\right)\right\}$.

It is clear that these are closed under addition. Can we now construct simple roots. It is not that difficult. These are those positive roots which cannot be written as the sum of positive roots. So in our example these would be $\alpha$ and $=-\alpha+\delta$. It's a general fact that every positive root can be written as a sum of simple roots, so it just remains to show that these are the only simple roots. This is a simple argument.

For $\mathfrak{s l}(3)$ there are two simple roots and three positives. There is only one simple root at the next level. I can get everything in the second plane from the "most negative" root and the simple roots, and this will generate everything.

Now we can formulate a general statement:
Let $\hat{\Pi}=\left\{\hat{\alpha} \in \hat{R}_{+} \mid \hat{\alpha}\right.$ is simple $\}$.
Theorem $1 \hat{\Pi}=\left\{\alpha_{0}, \alpha_{1}, \cdots, \alpha_{r}\right\}$, where the nonzero indexed roots are the simple roots for $R$ and $\alpha_{0}=-\theta+\delta$, where $\theta \in R_{+}$is the the highest (maximal) root.

You know about highest weights in representations; the same is true in the Lie algebra itself; there is a root such that any root $\alpha$ can be written as $\alpha=\theta-\sum n_{i} \alpha_{i}, n_{i} \in \mathbb{Z}_{+}$.

This follows easily if the Lie algebra is simple, and is still easy to check if it is semisimple.
For the record, let me say that $\theta$ is always a long root. In particular, we've chosen the inner product so that the square of the length of the long root is two, so that $(\theta, \theta)=2$.

Example $1 A_{n-1}: \quad R=\left\{e_{i}-e_{j}, i \neq j, i, j \in 1, \cdots, n\right\}$. The simple roots are $e_{i}-e_{i+1} . I$ claim that $\theta=e_{1}-e_{n}$ is the highest root. This is a pretty easy argument.

There are explicit formulas for the others, you can look them up if you like.
Here are some properties:

1. $\alpha_{0}, \cdots, \alpha_{r}$ are linearly independent.
2. Any $\hat{\alpha} \in \hat{R}_{+}$can be uniquely written as a sum $\hat{\alpha}=\sum_{i=0}^{r} n_{i} \alpha_{i}$ for $n_{i} \in \mathbb{Z}_{+}$.
3. Consider $\widehat{\widehat{\mathfrak{h}}_{k}^{\mathbb{R}}} / \mathbb{R} \delta$ and consider the positive Weyl alcove $C_{k}^{+}=\left\{\lambda \mid\left(\lambda+k \Lambda_{0}, \hat{\alpha}\right)>0 \forall \hat{\alpha} \in\right.$ $\left.\hat{R}_{+}\right\}$.

If you remember, the condition that all of these are zero gives us all of these hyperplanes so this is a connected component of the complement. So this is $\left\{\lambda \mid\left(\lambda+k \Lambda_{0}, \hat{\alpha_{i}}>0, i=0, \cdots, r\right\}\right.$.

Explicitly, this condition is $\left\{\lambda \mid\left(\lambda, \alpha_{i}\right)>0\right.$ for $i$ in $\left.1, \cdots, r\right\}$ and $-(\lambda, \theta)+k>0$. So $(\lambda, \theta)<k$.
Notice that this is nonempty only if $k$ is positive.

Corollary 1 Reflections $s_{i}, i=0, \ldots, r$ generate $\hat{W}$.

Let me remind you, the proof is the same as in the finite dimensional case. By using these reflections and conjugates I can map from one to another in a series of steps. Since I already know I can get any alcove, I can get something by conjugating one of my original roots by others.

It is not so easy to say in words, but it is the same proof as in finite dimensions.
So the whole infinite Weyl group is generated by $r+1$ reflections.
The final thing I wanted to define today is the Cartan matrix.
As usual, it is a matrix with entries $a_{i j}=\frac{2\left(\alpha_{i}, \alpha_{j}\right)}{\left(\alpha_{j}, \alpha_{j}\right)}$, now starting from 0 , not $r$.

Example $2 \bullet \widehat{\mathfrak{s l}(2)}, \alpha_{0}=-\alpha+\delta, \alpha_{1}=\alpha$. So $A=\left(\begin{array}{cc}2 & -2 \\ -2 & 2\end{array}\right)$. This is degenerate, which is to be expected since the form is only semidefinite.

- $\widehat{\mathfrak{s l}(n)}, \alpha_{0}=-\theta+\delta=e_{n}-e_{1}+\delta, \alpha_{i}=\alpha_{i}-\alpha_{i+1}$.

So $A=\left(\begin{array}{ccccc}2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & \ddots & \\ -1 & & & & 2\end{array}\right)$. You can see this is degenerate by summing all the rows.

Next time Dynkin diagrams.

